

Scuola Internazionale Superiore di Studi Avanzati



Vortices, Surfaces and Instantons

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Abstract

This thesis is based on some selected topics related to the Alday-Gaiotto-Tachikawa (AGT) duality, vortex counting and topological vertex. It begins with a review on necessary background materials for later chapters. Then we will study the nonabelian vortex counting problem and its relation with the strip amplitudes of topological vertex. After that we will demonstrate a degeneration phenomenon of instanton partition functions of quiver gauge theories and obtain the two-dimensional CFT dual of nonabelian vortices. These results will be generalized to instanton/vortex on orbifolds and the $\mathcal{N} = 1$ super Liouville theories in the following chapter.

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Introduction

Quantum field theory has proven to be a successful theoretical framework for the study of fundamental particles and condensed matter physics. However many of the most celebrated successes are in the perturbative regime which deals with weakly coupled fields. In order to extend the theory to the non-perturbative regime where fields are strongly coupled further effort is required. One way to approach this problem is to add more symmetries.

Supersymmetry is a candidate symmetry for the real world and a search for supersymmetry is one of the priorities of the Large Hadron Collider (LHC). Supersymmetric quantum field theories have beautiful mathematical structures and interesting physical features, because supersymmetry restricts the theories and simplifies quantum corrections of various observables to a great degree. For example, $\mathcal{N} = 4$ four dimensional super Yang-Mills theory is believed to be exactly super conformal. This theory is heavily constrained by symmetries and has been studied extensively in literature. Four dimensional $\mathcal{N} = 1$ gauge theories have much less symmetries and generally have quantum perturbative corrections beyond one loop, so it is extremely hard to compute in full generality. $\mathcal{N} = 2$ gauge theories are special in the sense that they have a proper number of super symmetries to make the theories to be neither too rigid nor too complicated.

A core ingredient of $\mathcal{N} \geq 2$ gauge theories is that their superalgebras can have nontrivial central charges. Physically, this means that there are states whose masses equal the central charges of the super algebras, so the masses are protected by the algebras and will not run along the renormalization flow. These special states are known as Bogomol'nyi-Prasad-Sommerfield (BPS) states. Another nice property of $\mathcal{N} = 2$ gauge theories is that the low energy effective action of an $\mathcal{N} = 2$ super QCD is determined by a single holomorphic function, the prepotential which only has one loop and nonperturbative instanton corrections.

In 1994, [1, 2] Seiberg and Witten solved the problem of calculating $\mathcal{N} = 2$ prepotentials by giving a geometrical explanation of BPS spectra as specific integrals over cycles of the so-called Seiberg-Witten curves. Furthermore in 1997, from the perspective of IIA string theory and M-theory, [3] Witten showed how to obtain the Seiberg-Witten curves of $U(N)$

linear/elliptic quiver gauge theories which are conformal or asymptotically free.

Seiberg-Witten solution offers an indirect way to calculate instanton contributions, and in 2002 Nekrasov [4] gave a direct calculation of instanton partition functions by using equivariant localization techniques. Subsequently, [5] Nekrasov and Okounkov proved that the semiclassical limit of Nekrasov partition functions reproduces Seiberg-Witten curves.

A detailed study on Seiberg-Witten curves of quiver gauge theories [6] inspired Gaiotto to generalize the S-duality of $SU(2)$ theories to a large class of the $\mathcal{N} = 2$ superconformal gauge theories and found the generalized superconformal quiver gauge theories. Based on these insights, in the summer of 2009, [7] Alday, Gaiotto and Tachikawa discovered an amazing correspondence which was later named after them.

The so-called Alday-Gaiotto-Tachikawa (AGT) correspondence is an exact duality between the $\mathcal{N} = 2$ four dimensional quiver gauge theories and the Liouville/Toda conformal theories on Riemann surfaces encoding the quiver structures of the former. One important proposal of the duality is that Nekrasov's instanton partition functions can be identified with certain conformal blocks in the Liouville theory. They also found that the one-loop corrections of partition functions can be identified with the Dorn-Otto-Zamolodchikov-Zamolodchikov (DOZZ) formulae of the Liouville theory.

In the past few years various aspects of the AGT duality have been studied and many profound results have been found. For example, it was found that the lowest degenerate state in Liouville theory can be identified as the simple surface operator of four dimensional gauge theories [8]. While the simple surface operators turned out to be related to the vortices solutions of two dimensional gauge theories and special Lagrangian submanifolds of local Calabi-Yau manifolds [9, 10, 11].

The vortex story can be traced back to [12], when Dorey found that the BPS spectrum of the mass deformed two-dimensional $\mathcal{N} = (2, 2)$ \mathbb{CP}^{N-1} sigma-model coincides with the BPS spectrum of the four-dimensional $\mathcal{N} = 2$ $SU(N)$ supersymmetric QCD. Later in [13], Hanany and Tong proved that this $4d/2d$ duality can be explained by the BPS spectra of nonabelian vortex solutions in a $(2+1)$ dimensional field theory. They also showed that in string theory framework this duality is natural and clear. A similar result was found in $(3+1)$ dimensions in [14].

Stimulated by these observations, nonabelian vortex became an active area of research. The moduli space of vortices on a Riemann surface was studied in [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28] and the moduli space of vortices on \mathbb{C}/\mathbb{Z}_p was studied in [29]. The partition functions of nonabelian vortices on \mathbb{C} were calculated in [11, 30, 31] and vortex partition functions on S^2 were calculated in [32, 33].

Due to the fact that string/M theory have a large class of correspondences/dualities and

quantum field theories can be generated by string/M theory in various ways, embedding quantum field theories into string/M theory may formulate clearer, sometime novel correspondences/dualities among different theories. In the context of string theory, linear quiver gauge theories have a geometrical realization as the low energy effective theories of D4-branes intersecting with NS5-branes [3], and instantons can be considered as $D0$ -branes inside the $D4$ -branes. Two dimensional supersymmetric vortex solutions can also be extracted from a NS5-D4-D2 brane system [13, 34]. It is interesting to notice that when mass parameters of quiver gauge theories take special values, the instanton partition functions will degenerate into simpler forms [31]. The degeneration phenomenon of quiver instanton partition functions supplies a method to reduce the vortex partition functions from that of instantons and also reveals information about surface operators of corresponding gauge theories.

Another geometrical realization of the gauge theories is the so-called geometric engineering [35, 36]. Actually the two geometrical realizations are equivalent [37] and the brane construction diagrams can be identified with their corresponding toric diagrams, which suggests a possible relation between the nonabelian vortices and topological string amplitudes on strips [9, 10, 11]. One can refer to [38, 39, 40, 41] for topological vertex formalism of topological string amplitudes.

My Past Research

My PhD research centered on this celebrated AGT duality. We mainly studied relations among the surface operators in four dimensional gauge theories, degenerate field insertions in two dimensional conformal field theories, vortex solutions of two dimensional gauge theories, and A-model topological string amplitudes on strips. We calculated the nonabelian vortex partition functions and found that they can be identified as certain fusion channels of correlation functions of multiple degenerate field insertions in two dimensional Liouville/Toda conformal field theories. We further found that the nonabelian vortex partition functions can also be identified as four dimensional limits of certain A-model topological string amplitudes on strips. An interesting instanton/vortex relation was proposed and the relation between multiple surface operators and vortices was studied in detail. The relations among surface operators, degenerate fields and vortex partition functions were generalized to the case of orbifold theories and $\mathcal{N} = 1$ super Liouville theories.

In [11], we showed that the vortex moduli space can be embedded as a holomorphic submanifold of the instanton moduli space of $\mathcal{N} = 2$ four dimensional gauge theories. Using equivariant localization techniques we calculated the vortex partition functions for these theo-

ries. On one side, we showed that the vortex partition functions coincide with the field theory limit of the topological vertex on the strip with boundary conditions corresponding to one dimensional young tableaux; on the other side, we re-summed the vertex partition functions in terms of generalized hypergeometric functions formulating their AGT dual description as interacting simple surface operators. Correspondingly, the topological open string amplitudes are reshuffled in formalism of q -deformed generalized hypergeometric functions, which satisfy appropriate finite difference equations.

In [31], we aimed at finding two dimensional CFT duals of the vortex partition functions. It turned out we found more than that. We found there is a one-to-one correspondence between the fusion channels of correlation functions of multiple (semi-)degenerate field insertions in Liouville/Toda conformal field theories and the four-dimensional limit of open topological string amplitudes on strips with generic boundary conditions. As a byproduct we identified the nonabelian vortex partition functions as certain degenerations of the quiver instanton partition functions, which, according to the AGT relation, are specific fusion channels of degenerate conformal blocks. Geometrically, we found that fusion channels in Liouville/Toda theories correspond to D2/D4-brane configurations of the associated four dimensional quiver gauge theories.

The AGT correspondence was generalized to the case of ALE instantons and $\mathcal{N} = 1$ super Liouville theory [42, 43, 44, 45, 46, 47, 48, 49, 50]. In [51], we studied AGT like relations between the orbifold vortex and super Liouville theories. We realized that the relation between the vortex partition functions and the quiver instanton partition functions found in [31] can be generalized to the orbifold case with suitable conditions. By the same logic, we could identify the orbifold vortex partition functions as certain fusion channels of $\mathcal{N} = 1$ super Liouville conformal blocks. A pleasant byproduct is a promising AGT relation between the four Ramond primary fields and a certain branch of the four dimensional ALE gauge theory.

I also used to work on matrix models and the wall-crossing formulae [52]. Based on conjectured—later proved in [53]—formulae of refined BPS partition functions we showed how to obtain matrix models corresponding to the refined BPS states partition functions of \mathbb{C}^3 , resolved conifolds and $\mathbb{C}^3/\mathbb{Z}_2$. The refinement of BPS partition functions is closely related to the refinement of the topological vertex. We discussed subtleties of both kinds of refinements.

Structure of The Thesis

The thesis is organized as follows. Chapter one is a brief review on the IIA brane and M-theory construction of the linear/elliptic quiver gauge theories, $\mathcal{N} = 2$ dualities, and the AGT

duality. These materials are essential for understanding the following chapters. Some useful formulae for instanton counting and topological vertex are provided in the appendices. A review of instanton counting is not included since it is too technical and there are already plenty of excellent reviews about it. Chapter two focuses on how to calculate the nonabelian vortex partition functions using equivariant localization techniques. We will inspect the hypergeometric structure of the vortex partition functions and the reduction of vortex partition functions from topological string amplitudes. Chapter three deals with the problem of the AGT like duality for vortices. After carefully studying instanton partition functions of linear quiver gauge theories, we will show that vortex partition functions can be obtained from a serial degenerations of instanton partition functions. Consequently, by using the standard AGT dictionary we can interpret the nonabelian vortices as a certain fusion channel of multiple insertions of lowest degenerate states of the Liouville theory. In chapter four the results of chapter two and three are generalized to the case of instantons/vortices on orbifolds and their conformal theory correspondences in the $\mathcal{N} = 1$ super Liouville theory. Chapter five presents conclusions and discussions.

Chapter 1

Some Tales of $\mathcal{N} = 2$ Gauge Theories

In this chapter we will briefly review IIA brane and M-theory constructions of the linear quiver gauge theories [3], Gaiotto's construction of the generalized quiver gauge theories [6], and the AGT duality [7]. The brane construction is the key for the results studied in the following chapters, and it is also the starting point of Gaiotto's construction. Gaiotto's results inspired the discovery of the remarkable AGT duality. The following Table 1.1 is the core of the AGT duality, which is partially copied from [7]. One of the tasks of this chapter is to make this table easier to understand.

Gauge theory	Liouville theory
Instanton partition function	Conformal blocks
One-loop part partition function	Product of DOZZ factors
Integral of $ Z_{\text{full}}^2 $	Liouville correlator

Table 1.1: The Dictionary of the Liouville/gauge theory, where Z_{full} is the full partition function.

1.1 The IIA Brane and M-theory Constructions of $\mathcal{N} = 2$ Theories

As discussed in the introduction chapter, embedding quantum field theories into string/M-theory may allow us to find new connections among different theories. In [3], Witten gave the type IIA string theory and M-theory brane constructions of the $\mathcal{N} = 2$ linear/elliptic quiver gauge theories of A_{N-1} type and solved these theories by giving explicitly their Seiberg-Witten curves. In this section we will briefly review this story.

Figure 1.1 is an example of the brane construction of the $\mathcal{N} = 2$ pure $SU(2)$ gauge theory and its quiver diagram. In the following we will use rectangles to denote flavor nodes and

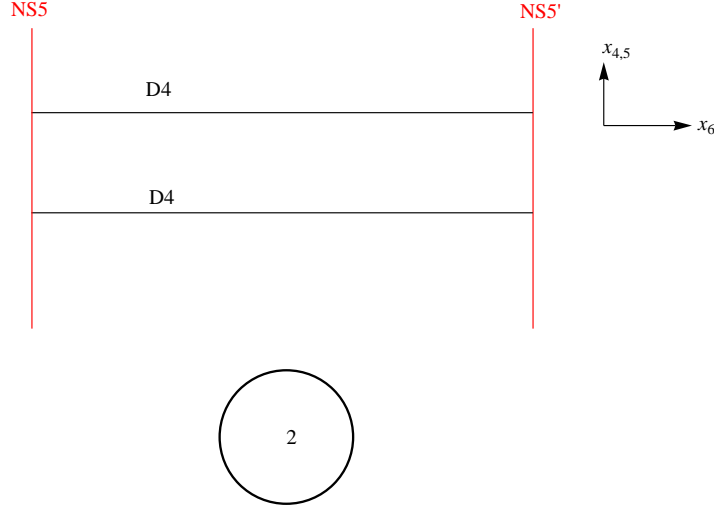


Figure 1.1: The IIA brane construction of the $\mathcal{N} = 2$ pure $SU(2)$ theory and its quiver diagram. The red lines denote the NS5-branes and the black lines denote the D4-branes. The quiver diagram for this theory has only one node, the $SU(2)$ gauge node.

circles to denote gauge nodes in all quiver diagrams. The number, N_f , inside a flavor node is the number of flavors and the number, N , inside a gauge node means the corresponding gauge group is $SU(N)$. The configuration of the branes is also shown in Table 1.2. The NS5-branes are infinite in six dimensions and intersect with the D4-branes in four dimensions—the four spacetime dimensions of the gauge theory. Since the NS5-branes are much heavier than the D4-branes, they can be considered as classical objects, and the gauge theory is the low energy effective field theory of the open strings attached on the D4-branes. The NS5-branes together with the D4-branes break the original ten dimensional $\mathcal{N} = 2$ supersymmetry into the four dimensional $\mathcal{N} = 2$ supersymmetry. Because the D4-branes is finite along the sixth dimension, the degrees of freedom along that direction can not appear in the low energy effective field theory, so finally we get a four dimensional $\mathcal{N} = 2$ pure $SU(2)$ gauge theory.

$brane \backslash dim$	0	1	2	3	4	5	6	7	8	9
NS5	○	○	○	○	○	○	×	×	×	×
D4	○	○	○	○	×	×	○	×	×	×

Table 1.2: The configuration of the IIA branes: ○ means parallel directions of branes; × means perpendicular directions of branes.

To solve the $\mathcal{N} = 2$ theories, it is convenient to use Seiberg-Witten curves. Let us explain how to obtain Seiberg-Witten curves of gauge theories from their brane construction. Let

us first introduce a complex coordinate $v = x_4 + ix_5$, which describes the location of the intersection of a D4-brane and an NS5-brane. For the pure $SU(2)$ theory there are two D4-branes, so we have v_1, v_2 . The mean value of v_1 and v_2 describes the motion of the center of mass of the two D4 branes and it is decoupled from the rest of the theory. While the relative value, $v_1 - v_2$ can be considered as the Coulomb branch parameter of the $SU(2)$ gauge theory.

We know that the gauge coupling constant g is related to the size of D4-branes along the sixth dimension. The explicit formula is

$$\frac{1}{g^2} = \frac{x_{L,6} - x_{R,6}}{g_s}, \quad (1.1.1)$$

where g_s is the string coupling constant and $x_{L,6}, x_{R,6}$ are the sixth coordinates of the left and right NS5-brane respectively. As a function of v , $x_{L,6}$ and $x_{R,6}$ describe how NS5-branes bend due to the existence of D4-branes. So $x_6(v)$ should vanish if the number of D4-branes on both sides of the NS5-brane are the same. The coupling constant can also be calculated perturbatively and have the form

$$\frac{8\pi^2}{g^2} = b \ln(|v|) + \text{const}, \quad (1.1.2)$$

where b is a group theory factor, which just depends on the representations of various fields. For the four dimensional $\mathcal{N} = 2$ $SU(2)$ gauge theory with N_f fundamental/antifundamental hypermultiplets and A adjoint hypermultiplets, $b = 2N - N_f - 2A$. Since we are only interested in field theories which are asymptotic free or superconformal, the b factor should be nonnegative.

When b is zero, the theory is superconformal and $x_6(v)$ should vanish which tells us that there should be equal numbers of D4-branes on both sides of the NS5-branes. Actually from this balancing requirement, we can get the constant b just by counting the number of branes. Two sufficient examples are $\mathcal{N} = 2$ $SU(N)$, $N_f = 2N$ and $\mathcal{N} = 2^*$ $SU(N)$ gauge theories. Let us first study the $\mathcal{N} = 2$ $SU(N)$, $N_f = 2N$ theory whose brane configuration is illustrated in Figure 1.2, where all hypermultiplets are in the fundamental/antifundamental representations. This is a balanced situation, and it satisfies $N_f = 2N$. Another balanced situation is when the hypermultiplet is in the adjoint representation (see Figure 1.3), and we have $A = N$. This is the so-called $\mathcal{N} = 2^*$ $SU(N)$ theory. Since the contributions of various fields to the β -function are linearly independent, we get the general formula $b = \alpha(2N - N_f - 2A)$, where α is a constant.

However in the existence of the so-called *theta*-term, the gauge coupling constant is only the real part of the complexified coupling constant. The imaginary part is the *theta*-term

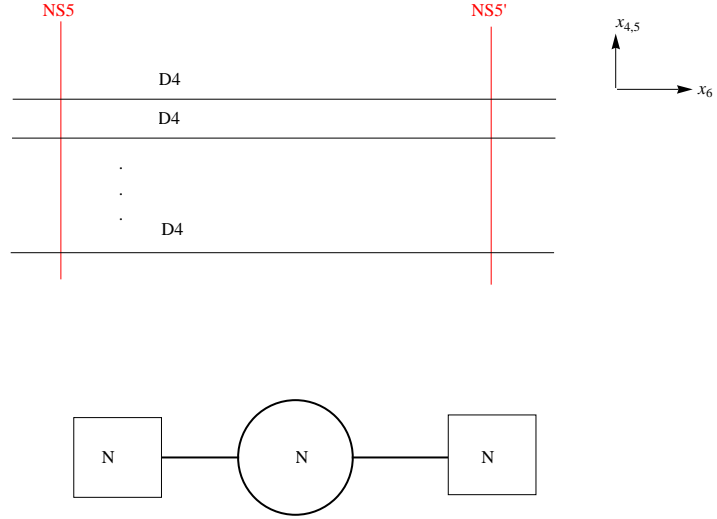


Figure 1.2: The IIA brane construction of the $\mathcal{N} = 2$ $SU(N)$, $N_f = 2N$ theory and its quiver diagram.

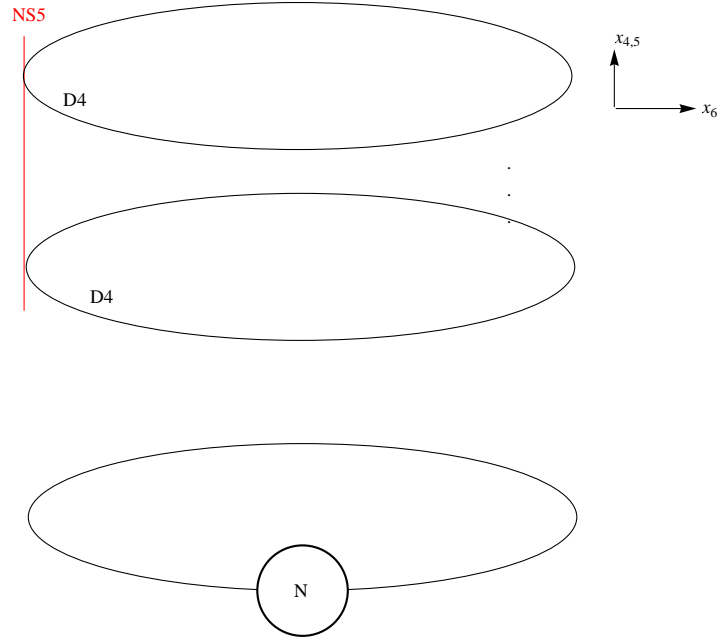


Figure 1.3: The IIA brane construction of the $\mathcal{N} = 2^*$ $SU(N)$ theory and its quiver diagram.

coefficient, which should take value in S^1 . This observation leads us to identify θ as the tenth coordinate of M-theory. In general, the complex coupling constant of gauge theory is defined as

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}. \quad (1.1.3)$$

Correspondingly in M-theory we can define the dimensionless complex coordinate as

$$s = \frac{x_6 + ix_{10}}{R}. \quad (1.1.4)$$

Since s is not single-valued, we will introduce $t = \exp(-s)$. Step by step, we are forced to go inside M-theory in order to study the four dimensional $\mathcal{N} = 2$ gauge theories. In M-theory, NS5-branes are M5-branes which do not wrap on the tenth dimension and D4-branes are M5-branes wrapping on the tenth dimension. What is more interesting is that the singularities at the intersection of NS5 and D4 branes disappear in M-theory. The reason is that, in M-theory, all the IIA branes turn out to be a single M5 which wraps on a smooth surface Σ which encodes the configuration of NS5 and D4 branes. From previous discussion we know that the configuration of branes are described by v and t and if we assume v and t are holomorphic coordinates then $\mathcal{N} = 2$ supersymmetry assures that Σ is a complex Riemann surface and can be described by a polynomial function $F(t, v) = 0$.

In the following we will see how to get the Seiberg-Witten curve for the pure $SU(2)$ gauge theory. For fixed v , the solution of $F(t, v) = 0$ should give the positions of the two NS5-branes, so $F(t, v)$ is a polynomial in t of degree two. Since there are no D4-branes at $x_6 \rightarrow \infty$ or $x_6 \rightarrow -\infty$, $F(0, v) = 0$ and $F(\infty, v) = 0$ should have no solution and $F(t, v) = t^2 + B(v)t + 1$. For fixed t , $F(t, v) = 0$ should give the positions of the D4-branes, which means $B(v)$ is a polynomial in v of degree two. Since the relative distance of the two D4-branes can be explained as the Coulomb branch parameter, we can write $B(v) = v^2 + u$. From the physical meaning of v , we know that u is the invariant Coulomb branch parameter. Finally, we get

$$t^2 + (v^2 + u)t + 1 = 0. \quad (1.1.5)$$

This is the Seiberg-Witten curve for the $\mathcal{N} = 2$ pure $SU(2)$ gauge theory, and the Seiberg-Witten differential is

$$\lambda = \frac{v}{t} dt. \quad (1.1.6)$$

This simple example can be generalized to more general situations [3]: more NS5-branes, higher rank gauge groups, multiple nodes of gauge groups and matter fields in other representations. Usually, quiver theories are called linear quiver theories if the corresponding

quiver diagrams look like an open necklace as shown in Figure 1.4. Otherwise if their quiver diagrams look like a closed necklace, they are called elliptic quiver theories (Figure 1.5).

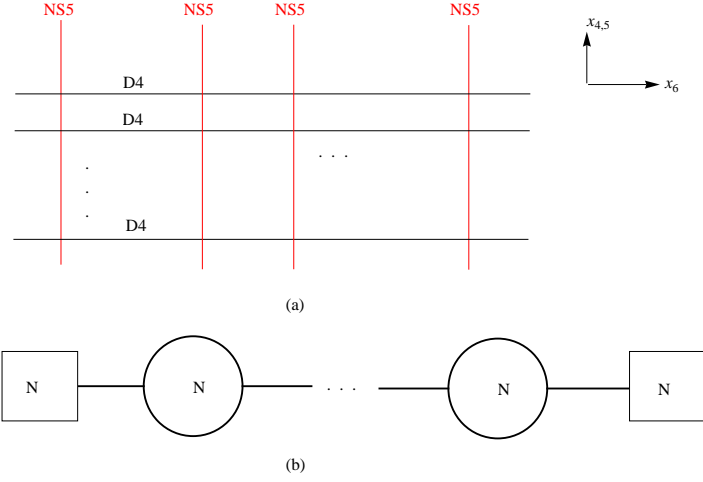


Figure 1.4: The IIA brane construction of quiver theories with multiple gauge nodes. (a) is the brane configuration of a generic linear quiver theory. (b) is its quiver diagram.

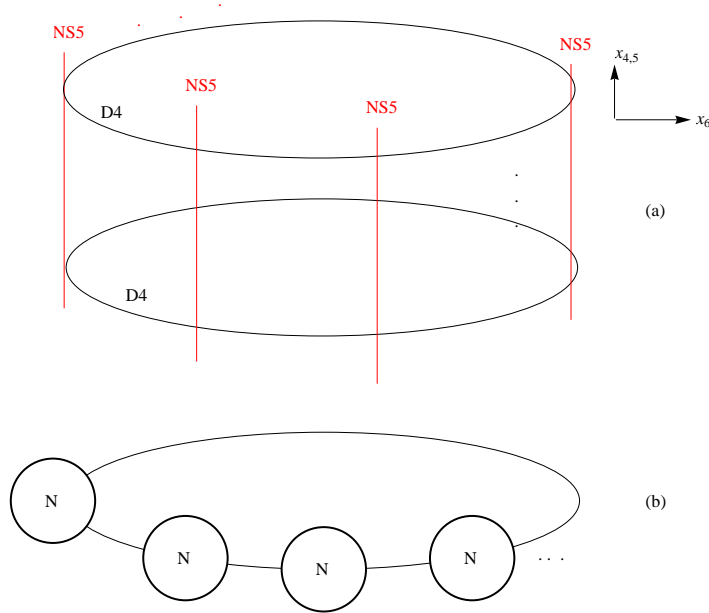


Figure 1.5: The IIA brane construction of quiver theories with multiple gauge nodes. (a) is the brane construction of a generic elliptic quiver theory. (b) is its quiver diagram.

1.2 The $\mathcal{N} = 2$ Dualities and Generalized Quiver Theories

A generalized quiver theory [3, 6] or a theory of the class \mathcal{S} [54, 55] can be constructed by compactifying and partially twisting the $\mathcal{N} = (2, 0), d = 6$ theories on a Riemann surface C . In general the theory is denoted as $S[C, \mathfrak{g}, D]$, where C is a compact Riemann surface, with n punctures at the points z_1, z_2, \dots, z_n , and \mathfrak{g} is a Lie algebra of the ADE type. D is a set of half-BPS codimension-2 defects placed at the punctures. Notice that the Seiberg-Witten curve of the theory is a multiple cover of the curve C . The genus of the Seiberg-Witten curve is determined by the rank of the gauge group, while the genus of C is determined by the number of loops of the corresponding generalized quiver diagram. We will start from the simplest $SU(2)$ theories and then go to the $SU(3)$ theories. These two examples are simple but illuminating enough for the inspection of general features of the theories. Finally, we will study the $SU(N)$ theories, focusing on the classification of punctures.

1.2.1 $SU(2)$ Theories

It is known that the $SU(N_f)$ flavor symmetry supported by N_f hypermultiplets can be enhanced to $SO(2N_f)$ or $Sp(2N_f)$ if the hypermultiplets are in pseudoreal or real representations of the gauge group [2]. For the $\mathcal{N} = 2$ $SU(2)$ gauge theory with $SU(4)$ flavor group, the enhanced flavor group is $SO(8)$. It was also found in [2] that this theory has an S-duality group $SL(2, \mathbb{Z})$ which acts in the same way on the three eight-dimensional representations of $SO(8)$ and the three even spin structures on the torus of the Seiberg-Witten curve.

To further study the $\mathcal{N} = 2$ duality, we should study further the flavor symmetry. It is interesting to follow the details of the transformation of the three eight-dimensional representations of the $SO(8)$ flavor symmetry [6]. The three eight-dimensional representations of $SO(8)$ are denoted as $8_v, 8_s, 8_c$, which can be rewritten as

$$\begin{aligned} 8_v &= (2_a \otimes 2_b) \oplus (2_c \otimes 2_d) , \\ 8_s &= (2_a \otimes 2_c) \oplus (2_b \otimes 2_d) , \\ 8_c &= (2_a \otimes 2_d) \oplus (2_b \otimes 2_c) , \end{aligned} \tag{1.2.1}$$

where $2_a, 2_b, 2_c, 2_d$ are fundamental representations of the four $SU(2)$ flavor groups $SU(2)_a, SU(2)_b, SU(2)_c, SU(2)_d$ respectively. Since each $SU(2)$ corresponds to a simple root of $SO(8)$, $SU(2)_a$ can be considered as the simple root at the center of the Dynkin diagram of $\mathfrak{so}(8)$, while the other $SU(2)$ groups are those surrounding the center. The permutation of the three surrounding simple roots, which is well-known as $SO(8)$ triality, corresponds to the permutation of the three eight-dimensional representations (1.2.1).

Another important fact is that H (the upper half plane), the parameter space of the gauge coupling, modulo the duality group and $\mathcal{M}_{0,4}$, the complex structure moduli space of a sphere with four identical punctures, are the same, that is $\mathcal{M}_{0,4} = H/\mathrm{SL}(2, \mathbb{Z})$. This suggests us to map the four $SU(2)$ flavor groups to the four identical punctures on a sphere and the $SU(2)$ gauge group is mapped to a tube connecting the four punctures. (Figure 1.6(b)). We will see later in this subsection that these punctures are poles of the Seiberg-Witten differential forms and in the canonical form, (1.2.4), these poles are of the same order. Since the S-duality group of the $SU(2)$, $N_f = 4$ theory permutes the four $SU(2)$ flavor groups in the same way as it permutes the four punctures on the sphere, it was proposed in [6] that the gauge theory S-duality group of the $SU(2)$, $N_f = 4$ theory can be identified with the fundamental group of the moduli space of the four-point punctured sphere.

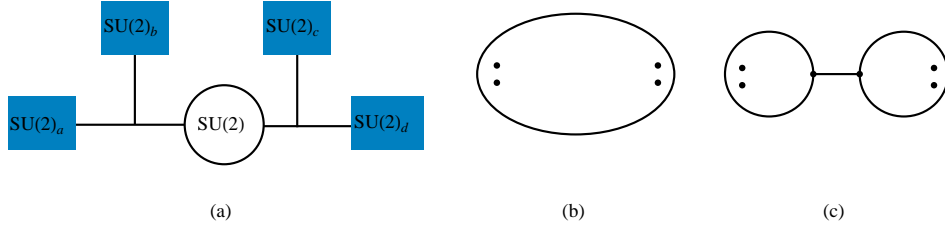


Figure 1.6: In part (a), the blue rectangles denote the $SU(2)$ flavor groups. In a generalized quiver diagram, group factors are usually written explicitly as shown here. But sometimes for the principle of simplicity, if there is no confusion we will omit the group factors written inside the nodes. The four-punctured sphere in (b) encodes the geometry of the gauge coupling parameter space of (a). Part (c) means in the weak coupling limit the gauge tube in (b) becomes a line. The weak coupling limit also give us a pants decomposition of (b) and this kind of decomposition will be referred frequently in the following diagrams.

We can draw a generalized quiver diagram for the theory as shown in Figure 1.6. Notice that in a generalized quiver diagram a flavor group is an *external* node, while a gauge group is an *internal* node. This is a general rule for superconformal linear quiver gauge theories and it is related to the AGT duality which will be discussed in the next section. One should also notice that a generalized quiver diagram is different from an (ordinary) quiver diagram which we have discussed in the previous section. For example in a generalized quiver diagram of $SU(2)$ theories, the flavor nodes are connected to the gauge nodes by trivalent vertices, while in an ordinary quiver diagram, the flavor nodes are always connected to the gauge nodes directly. Meanwhile, the flavor node coming from a bifundamental hypermultiplet is apparent in a generalized quiver diagram (Figure 1.7(c)) but implicit in an (ordinary) quiver diagram. The quiver diagrams are drawn in the “generalized” way in order to make the flavor groups more clear, since it is of vital importance in the study of S-dualities of $\mathcal{N} = 2$ theories.

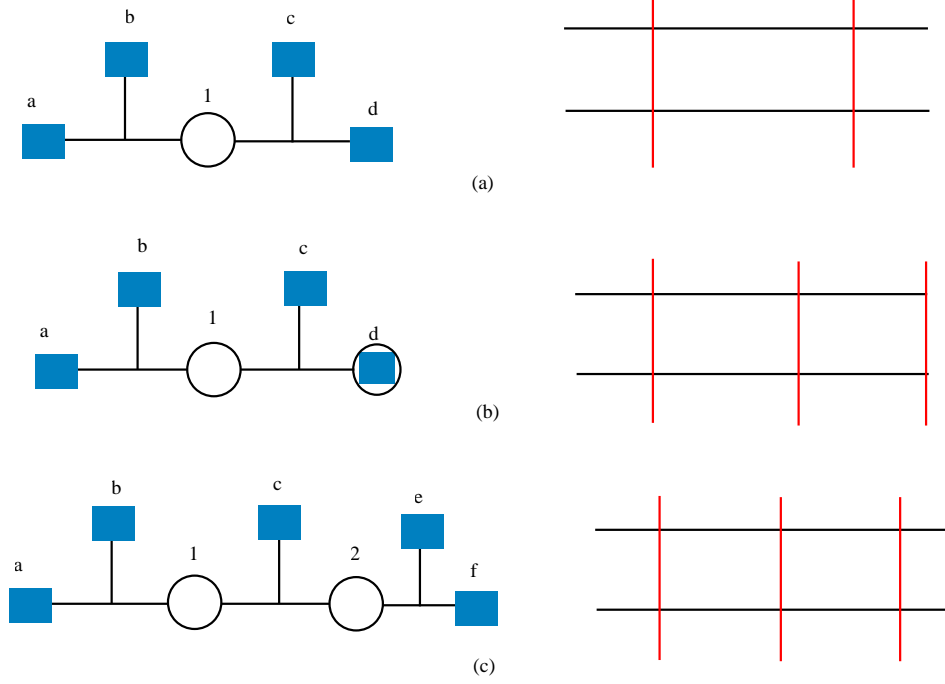


Figure 1.7: The IIA brane configurations and generalized quiver diagrams of gauging one flavor group. Rectangle-and-elliptic nodes means weakly gauging the flavor group. The numbers adjacent to nodes are labels of the nodes. Each node represents an $SU(2)$ group, which is implicit since no confusion will arise.

It is interesting to see how the quiver diagram changes when we gauge one of the flavor groups. In Figure 1.7(b) after gauging the $SU(2)_d$ flavor group, we get a two-node linear quiver gauge theory which is not superconformal, and the brane construction is shown on the left of the quiver diagram. If we add two fundamental hypermultiplets which are minimally coupled to the newly gauged $SU(2)_d$, the theory is superconformal and its quiver diagram and brane construction are shown in Figure 1.7(c).

We can transform Figure 1.7(a) to Figure 1.7(c) following another procedure: (1) we gauge the $SU(2)_d$ flavor group; (2) we construct another gauge theory which is one $SU(2)_3$ with two antifundamental hypermultiplets; (3) we identify the gauge group $SU(2)_d$ and $SU(2)_3$. Finally, we get a two-node superconformal linear quiver gauge theory. If the theory in step (2) is constructed by gauging a flavor $SU(2)_3$ group, this procedure is also called *gauging the diagonal subgroup* of the two flavor groups $SU(2)_d$, and $SU(2)_3$, as illustrated in Figure 1.8. The corresponding manipulation of punctures on Riemann surfaces (Figure 1.9) indicates that diagonal gauging corresponds to gluing two spheres through two punctures.

Now we want to find out the S-duality group for this extended quiver diagram. If the

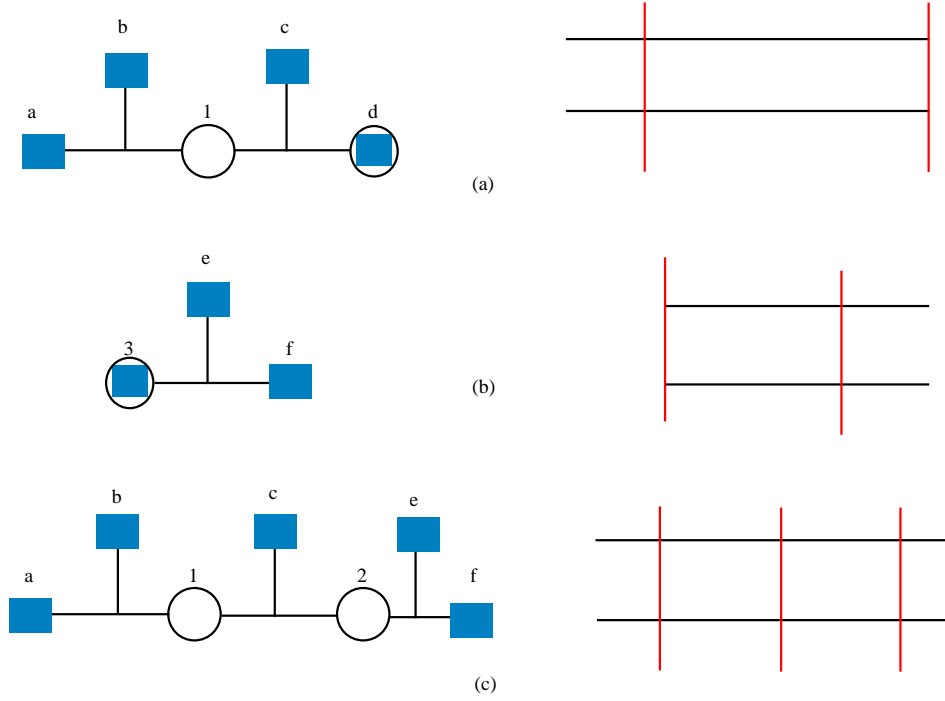


Figure 1.8: Diagonal gauging one flavor group and its brane configurations.

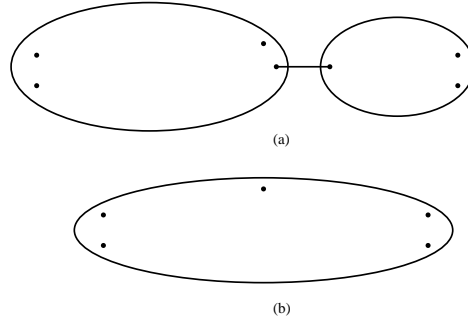


Figure 1.9: Diagonal gauging corresponds gluing of Riemann surfaces.

$SU(2)_2$ (Figure 1.8(c)) is weakly gauged, it is natural to expect that the S-duality group for the $SU(2)_1$ is still valid. As is clear in Figure 1.10, the S-duality operations of exchanging b and c , then a and 2 transform the original linear quiver theory into other linear quiver theories. The corresponding pants decomposition of Riemann surfaces are shown in Figure 1.11.

We have shown that the S-duality group works for the two-node¹ quiver theories and there are not many new things coming out. What is really new is the case of the three-node quiver

¹In general, when we say n -node quiver theories, we mean a quiver theory with n gauge nodes.

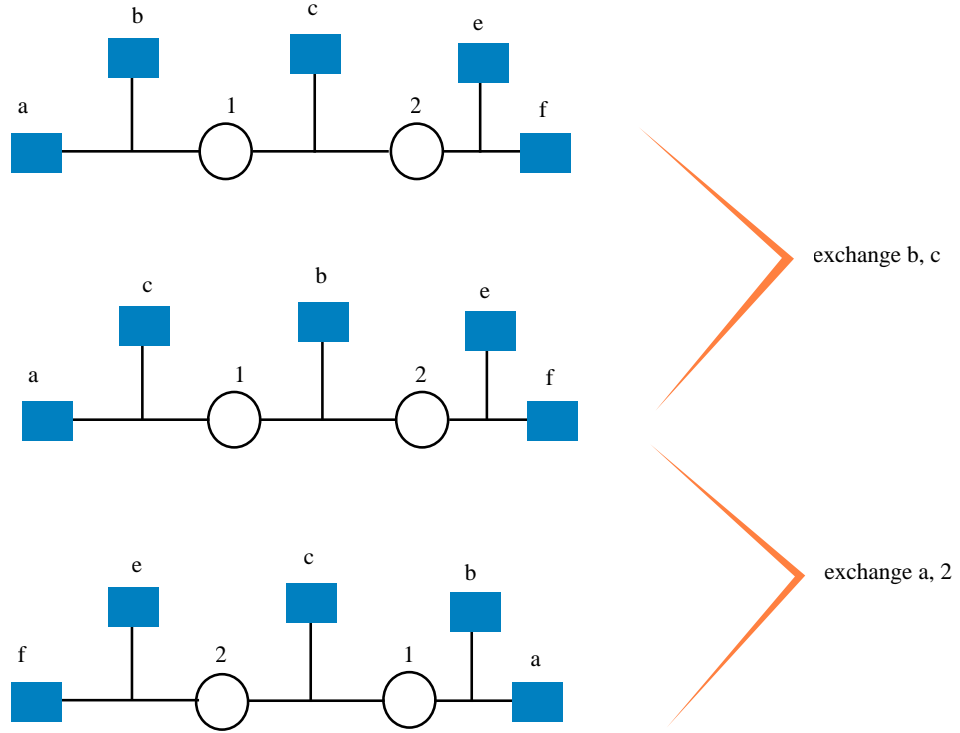


Figure 1.10: The S-duality transformations of the two node $SU(2)$ quiver theory from the point of view of quiver diagrams.

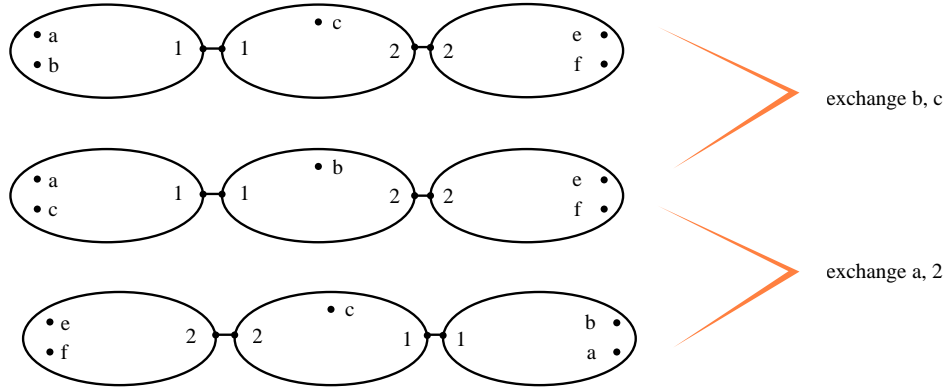


Figure 1.11: The S-duality transformations of the two node $SU(2)$ quiver theory from the point of view of punctured Riemann surfaces. Gauge groups are tubes connecting three-point punctured spheres and in the weak coupling limit, the tubes become lines.

theories. If we exchange the node a with the node 2 or node 3 (Figure 1.10), there is nothing new: a linear quiver theory is turned into another linear quiver theory. While if we exchange

the node c with the node 3 (Figure 1.12) what we get is not a linear quiver gauge theory, but a so-called generalized quiver gauge theory. The corresponding pants decomposition of Riemann surfaces are shown in Figure 1.13. This theory has no known IIA brane construction², which is one of the reasons it is called a “generalized” quiver gauge theory.

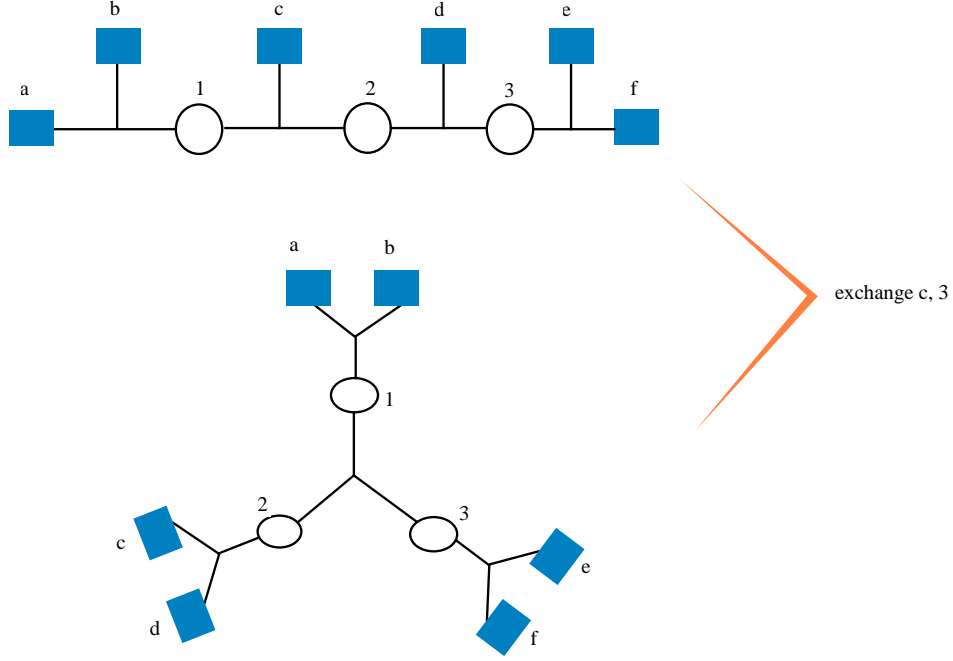


Figure 1.12: S-duality transformations that turn linear quiver theories into a generalized quiver theories from the point of view of quiver diagrams.

It was observed by Gaiotto [6] that in order to make flavor symmetries apparent in Seiberg-Witten curves, one should execute the following transformations: firstly we use the translation invariance of v to shift away the linear coefficient of v , secondly we do fractional linear transformations on the coordinates.

It is known that the Seiberg-Witten curve for the superconformal $SU(2)$ n -node linear quiver gauge theory is³

$$\begin{aligned} \Delta_{n+1}(t)v^2 &= M_{n+1}(t)v + U_{n+1}(t), \\ \Delta_{n+1}(t) &:= \prod_{a=0}^n (t - t_a). \end{aligned} \tag{1.2.2}$$

²In the strong coupling limit of IIA string theory, the M-theory, the theory has a clear description.

³It is a rule of thumb that in a Seiberg-Witten curve a capital letter denotes a polynomial and the subscript of the capital letter denotes the degree of the polynomial.

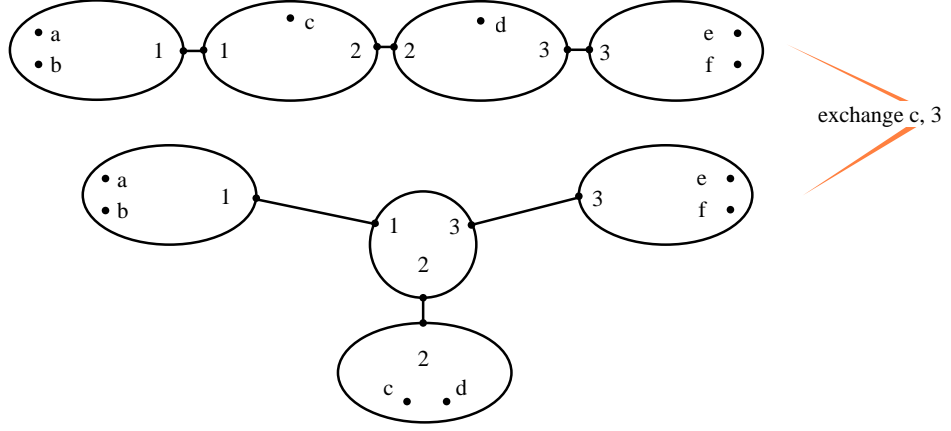


Figure 1.13: S-duality transformations that turn linear quiver theories into generalized quiver theories from the point of view of punctured Riemann surfaces.

So $M_{n+1}(t)$ and $U_{n+1}(t)$ in (1.2.3) are polynomials of degree $n + 1$ in t . When $t \rightarrow t_a$, $\Delta_{n+1}(t)$ has a zero of order one, then we can expand above formula around this zero point as

$$\begin{aligned}
 v &\simeq \frac{M_{n+1}(t_a)}{2\Delta_{n+1}(t_a)} \left(1 \pm \left(1 + \frac{2\Delta_{n+1}(t)U_{n+1}(t_a)}{M_{n+1}^2(t_a)} + \mathcal{O}((t-t_a)^2) \right) \right), \\
 v_+ &\simeq \frac{M_{n+1}(t_a)}{2\Delta_{n+1}(t_a)} + \mathcal{O}(1), \\
 v_- &\simeq \frac{-U_{n+1}(t_a)}{M_{n+1}(t_a)} + \mathcal{O}((t-t_a)).
 \end{aligned} \tag{1.2.3}$$

We see that between the two roots of v , only v_+ has a simple pole at t_a . At 0 and ∞ , v_{\pm} are constant terms, so the Seiberg-Witten differential λ has two simple poles at 0 and ∞ , while at the $n + 1$ zeros of $\Delta_{n+1}(t)$, λ has only one simple pole. Using the translation invariance of v , we get

$$\Delta_{n+1}^2(t)v^2 = \frac{M_{n+1}^2(t)}{4} + \frac{U_{n+1}(t)}{\Delta_{n+1}(t)}.$$

A nice feature of this shift is that now $\lambda = \frac{v}{t}dt$ has two simple poles at 0, ∞ , and $n+1$ zeros of $\Delta_{n+1}(t)$, and the two residues at each point are of the same absolute value but opposite signs. The residues of λ are interpreted as masses of hypermultiplets and charges of the Cartan subalgebra of the corresponding flavor group. A further coordinate change is that at first we set $v = xt$, and then do an $SL(2, \mathbb{C})$ transformation of x and t . Since under the fraction linear transformation $t \rightarrow \frac{az+b}{cz+d}$ and $\lambda = xdt$ is invariant, the transformation of x is

$x \longrightarrow x(cz + d)^2$. After these procedures, the Seiberg-Witten curve is

$$x^2 = \frac{P_{2n+2}(z)}{\Delta_{n+3}(z)^2} := \phi_2(z). \quad (1.2.4)$$

Now the Seiberg-Witten differential is $\lambda = xdz$, which has two simple poles at the $n + 3$ zeros of $\Delta_{n+3}(z)$, and this form of the Seiberg-Witten curve is called the canonical form [6]. Notice that, in these coordinates, λ is regular at ∞ .

1.2.2 $SU(3)$ Theories

The $SU(3)$ linear quiver theory with three fundamental and three antifundamental hypermultiplets is superconformal and its IIA brane construction is shown in Figure 1.14.b. Since the fundamental representation of $SU(3)$ is complex, the flavor symmetry can not be enhanced to a larger group. For matter fields in the fundamental/antifundamental representation, the flavor group is $U(3) = SU(3) \times U(1)$, so there are two $SU(3)$ and two $U(1)$ flavor groups in the quiver diagram (Figure 1.14.a). Recall that for $SU(2)$ gauge theories, $SU(2)$ flavor groups of quiver diagrams are identified with identical punctures on spheres, so it is natural to propose that for the $SU(3)$, $N_f = 6$ gauge theory the corresponding Riemann surface should be the form shown in Figure 1.14.c. The proposition is supported by two facts. One fact is that the moduli space of gauge coupling of the $SU(3)$, $N_f = 6$ gauge theory is the same as $\mathcal{M}_{0,(2,2)}$, which is also the moduli space of complex structures of a genus zero Riemann surface with four two-typed punctures. The other fact is that in the canonical form of the Seiberg-Witten curve, (1.2.9), the third order differential form $\phi_3 dt^3$ has two types of poles at these punctures. Let us call these punctures $U(1)$ and $SU(3)$ type punctures for convenience.

We also propose that the S-duality groups of $SU(3)$ gauge theories permutes punctures on the corresponding Riemann surfaces. In the following we want to know how much we can learn from this assumption. To study the S-duality group of the $SU(3)$, $N_f = 6$ theory, it is safe to start from the weak coupling region where the IIA brane construction is valid. In that case, (Figure 1.14(c)) the punctured sphere is slender and the two punctures close to each other are of different types. When the coupling become stronger, the punctures get closer to each other and the original slender sphere turns fat and looks like a round sphere. When the coupling approaches infinity, the sphere becomes slender again, but now the two punctures close to each other are of the same type⁴. The lower part of the sphere looks quite the same as half of the sphere of the $SU(2)$ linear quiver theory (Figure 1.6). So it is reasonable to conjecture that these two flavor groups couple to an $SU(2)$ gauge group. We

⁴If at the infinite strong coupling limit, the punctures close to each other are of different types, we get the original weakly coupled theory.

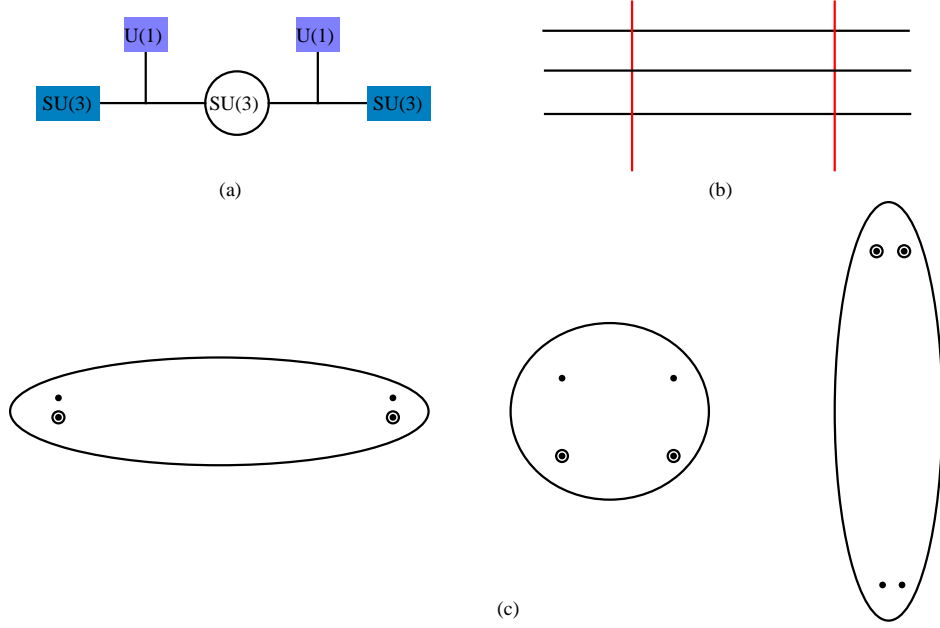


Figure 1.14: (a) is the generalized quiver diagram of the $SU(3)$, $N_f = 6$ theory. Since there are two types of flavor groups, the flavor nodes are discriminated by two colors. Correspondingly, there are two types of punctures on the sphere. (b) is its IIA brane construction. (c) shows how the S-duality moves the punctures on the sphere.

can not simply exchange the flavor factors as shown in Figure 1.15(b), because this simple-minded quiver has too many flavors and it is not superconformal. But it is sure that the $SU(2)$ somehow couples to two $SU(3)$ flavor groups. We can add an $SU(3)$ gauge node, which will not change the punctured sphere, and we find that the generalized quiver diagram (Figure 1.15(c)) is superconformal. Consequently, the rank of the theory is increased by one and the number of the flavors is decreased by one. If we simply replace the $SU(3)$ gauge node with an $SU(2)$ one (Figure 1.15(d)), we will meet the problem in the previous case: the new $SU(2)$ node is not conformal. It seems we need to generalize the generalized quiver diagrams.

The quiver in Figure 1.16 has two $SU(2)$ groups so that the ranks of the proposed dual pairs are the same. The two gauge nodes are encircled by a larger node which means the two $SU(2)$ groups interact with each other and they couple together with the two $SU(3)$ flavor groups. This theory has rank two and six flavors, so it may be superconformal. If this is the final story, the upper half of the sphere with two $SU(3)$ type punctures corresponds to the newly introduced node with two $SU(3)$ flavor groups. They can be considered coming from a complete theory whose groups have Dynkin diagrams including elements in Figure 1.16(b).

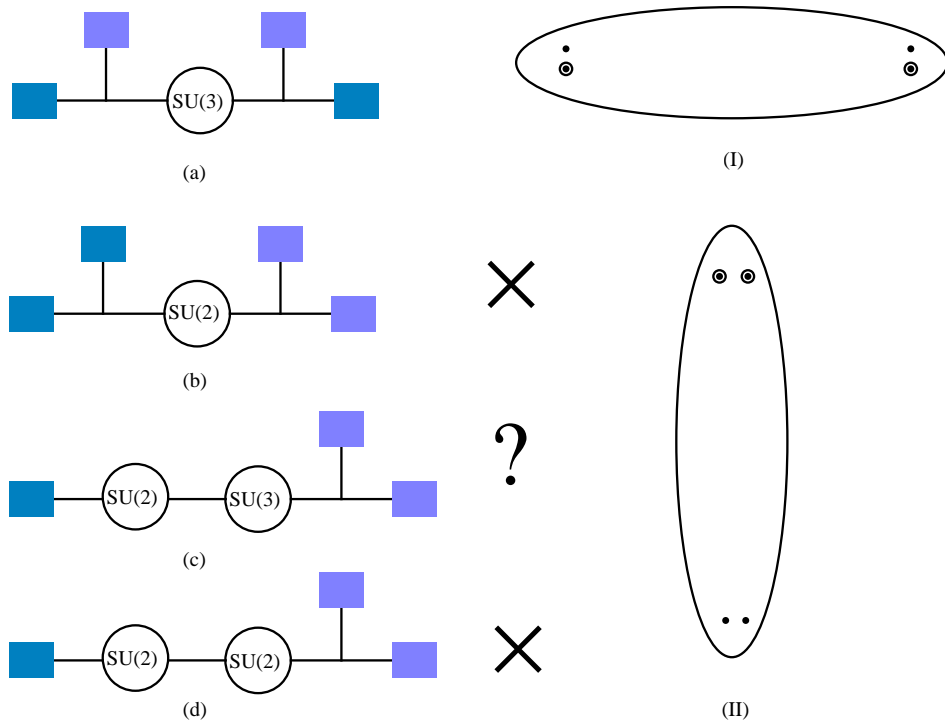


Figure 1.15: How far can we go by identifying $\mathcal{M}_{0,(2,2)}$ as the moduli space of marginal deformations (the parameter space of coupling constants modulo the S-duality group).

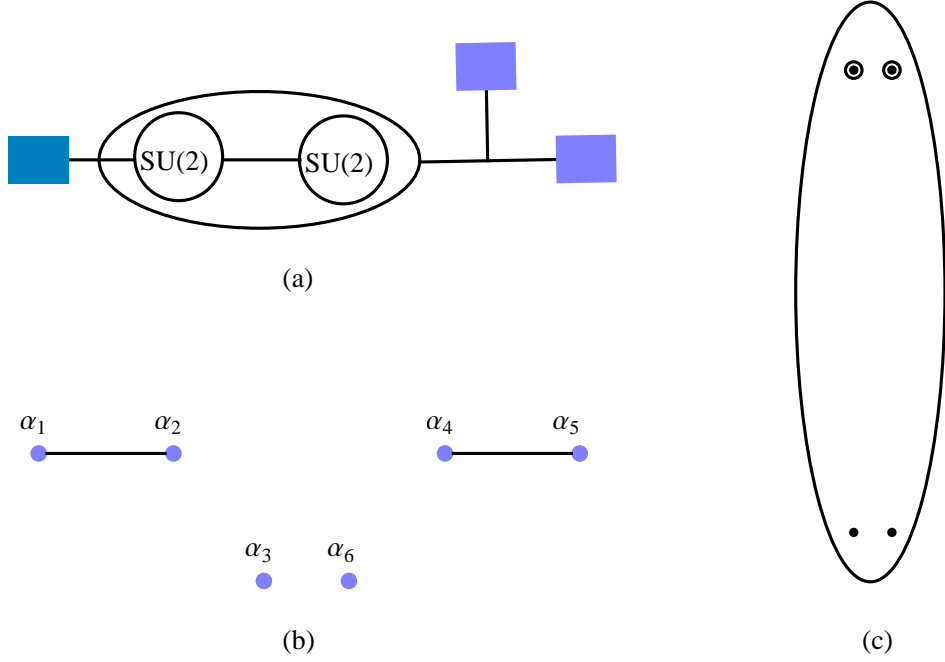


Figure 1.16: Another conjecture of the S-dual of the $SU(3)$, $N_f = 6$ theory. α_1, α_2 and α_4, α_5 correspond to the simple roots of the two $SU(3)$ flavor groups. α_3 and α_6 are the simple roots of the two $SU(2)$ gauge groups.

Actually we are approaching the ideal answer, which was discovered by Argyres and Seiberg in 2007 [56]. Their result is that the S-dual of the extremely strong coupling $SU(3)$, $N_f = 6$ theory has two parts. One part is an $SU(2)$ gauge theory coupled with one fundamental hypermultiplets. The other part is a rank one superconformal theory with flavor group E_6 . The two parts interact by gauging the $SU(2)$ in the maximal subgroup $SU(2) \times SU(6) \subset E_6$. In the Dynkin diagram (Figure 1.17), this corresponds to group α_1 to α_5 as the simple roots of $SU(6)$ and α_6 as the simple root of $SU(2)$. The two $SU(3)$ flavor groups come from the decomposition of $SU(6)$ as $SU(3) \times SU(3) \times U(1) \subset SU(6)$.

Another decomposition of E_6 is $SU(3) \times SU(3) \times SU(3)$, which corresponds to group $(\alpha_1, \alpha_2), (\alpha_4, \alpha_5), (\alpha_3, \alpha_6)$ as the simple roots of three $SU(3)$ groups (Figure 1.17). This more symmetric decomposition was used by Gaiotto to draw a generalized quiver for the $SU(3)$ theories. However the style of this quiver diagram is somehow a mixture of the ordinary and the generalized one. The single fundamental hypermultiplet is of the ordinary style, while the two $SU(3)$ groups are of the generalized style. Moreover, in this diagram, the two flavor groups corresponding to the two $U(1)$ type punctures are not clear. A comment here about the single fundamental hypermultiplet in Figure 1.18(a) is that the $SU(2)$ gauge group in Figure 1.18(a) can be considered coming from gauging an $SU(2)$ part of the tensor product

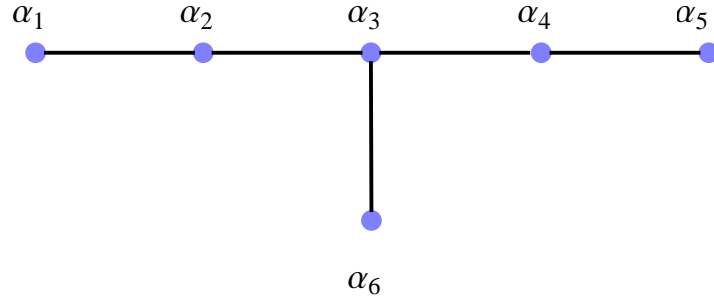


Figure 1.17: The Dynkin diagram of E_6 .

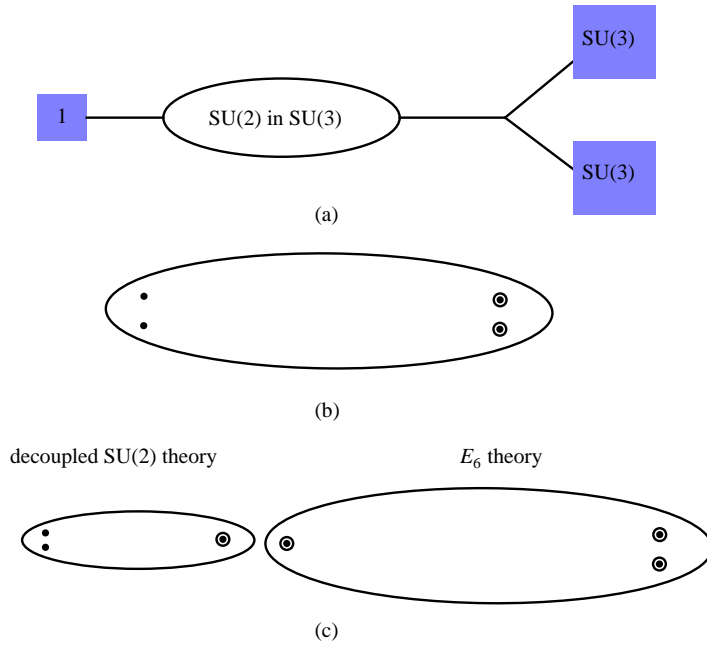


Figure 1.18: The Argyres-Seiberg duality. (a) is the generalized quiver diagram of the S-dual of the extremely strong coupling $SU(3)$, $N_f = 6$ theory. (b) is the moduli space of marginal deformations. (c) is the decoupling limit of (b).

of the two $U(1)$ fundamental hypermultiplets in Figure 1.18(b), while the left fundamental hypermultiplet in Figure 1.18(a) comes from the trivial part of the tensor product. This is the reason why in Figure 1.18(a) the gauge node is written as $SU(2)$ in $SU(3)$.

We can use the Seiberg-Witten curve to check the Argyres-Seiberg duality. Recall that the canonical Seiberg-Witten curve for the $SU(3)$, $N_f = 6$ theory is

$$\begin{aligned} x^3 &= \phi_2(t)x + \phi_3(t), \\ \phi_2 &= \frac{u^{(2)}}{(t-1)(t-t_1)t}, \\ \phi_3 &= \frac{u^{(3)}}{(t-1)(t-t_1)t^2}. \end{aligned} \tag{1.2.5}$$

It follows that the table of the pole structures of the two differential forms is

	0	1	t	∞
$\phi_2 dt^2$	1	1	1	1
$\phi_3 dt^3$	2	1	1	2

Reading column by column, we see that each puncture is specified by three parameters (t_a, p_2, p_3) , where t_a is the position of the puncture, and p_2, p_3 are the poles' orders of $\phi_2 dt^2$, $\phi_3 dt^3$ respectively.

At the strong coupling limit $t_1 = 1$, if we turn off $u^{(2)}$, the Seiberg-Witten curve becomes that of the E_6 theory,

$$x^3 = \frac{u^{(3)}}{(t-1)^2 t^2}. \tag{1.2.6}$$

On the other hand if we turn off $u^{(3)}$ at the strong coupling limit, we get a Seiberg-Witten curve from an $SU(2)$ theory. This decoupling limit is an essential check for the Argyres-Seiberg duality. In general the canonical Seiberg-Witten curve of an $SU(3)$ $(n-1)$ -node linear quiver theory is (1.2.5) with

$$\phi_2 = \frac{U_{n-1}^{(2)}(t)}{\prod_{a=0}^n (t-t_a)t}, \tag{1.2.7}$$

$$\phi_3 = \frac{U_{n-1}^{(3)}(t)}{\prod_{a=0}^n (t-t_a)t^2}. \tag{1.2.8}$$

Consequently, the table of the pole structures is

	0	$\{t_a\}_{a=0}^n$	∞
$\phi_2 dt^2$	1	1	1
$\phi_3 dt^3$	2	1	2
flavor	$SU(3)$	$U(1)$	$SU(3)$

(1.2.9)

where the flavor groups of the punctures are given in the bottom row.

1.2.3 The $SU(N)$ Generalization

For a general $SU(N)$ gauge theory, there is no Argyres-Seiberg like duality. The main tools we will use to study higher rank theories are their Seiberg-Witten curves,

$$x^N = \sum_{i=2}^N x^{N-i} \phi_i(z), \quad (1.2.10)$$

where $\phi_i dz^i$ are degree i differentials, (x, z) are local coordinates of $T^*C_{g,n}$ and $C_{g,n}$ is an n -punctured Riemann surface of genus g which equals the number of loops of the corresponding quiver diagram. The Seiberg-Witten differential is the canonical form, $\lambda = x dz$, of $C_{g,n}$.

The most important lesson we will learn in this section is the relations among the pole structures of $x dz$, ϕ_i and the flavor symmetries at each puncture. We already know that the $SU(2)$ theories are extremely simple, since there is only one kind of puncture with only one flavor symmetry, $SU(2)$. While the $SU(3)$ theories have two types of punctures with flavor groups $U(1)$ and $U(3)$. Because the residues of $x dz$ are masses of BPS states, the pole structures of $x dz$ are directly related with flavor symmetries. It follows that the general rules can be read from the Seiberg-Witten curves.

For an $SU(N)$ theory, x is an N -multiple cover of z , so for a given z there are N solutions of x , and at most N different masses. The multiplicities of masses can be described by a partition of N , which can also be illustrated by a two-dimensional Young tableau with N boxes. For such a Young tableau, Y , we require that the heights, h_i , of columns represent the numbers of identical masses. The flavor symmetry group at z is therefore $S(\prod U(N_{h_i}))$, where N_{h_i} is the number of columns of height h_i . The S here means we will only pick out the “special” part of the product group, since all flavor symmetries under consideration are irreducible.

If $Y = \{h_i\}_{i=1}^l$ and its dual partition $Y^t = \{q_i\}_{i=1}^{l'}$, for the box at the position $(i, j) \in Y$, we can assign a number $n_{i,j} = i + \sum_{k=1}^{j-1} q_k$ to it. It was shown in [6] that the pole’s order of

$\phi_{n_{i,j}}$ is $p_{n_{i,j}} = n_{i,j} - j$.

If $Y = \{N\}$, the flavor symmetry is $S(U(1))$, which is defined to be zero. This means that if all the N masses are the same, there is no flavor symmetry. A puncture is called a *basic puncture* if its corresponding Young tableau is $Y = \{N - 1, 1\}$ and the flavor symmetry is $S(U(1) \times U(1)) = U(1)$. Another important example is when $h_i \equiv 1$, the flavor symmetry is $S(\prod_{i=1}^N U(1)) = SU(N)$. This type of puncture is called a *full puncture*. Notice that a flavor symmetry is not uniquely specified by a Young tableau. For example all partitions of the form $Y = \{h_1, h_2\}$ with $h_1 > h_2$ give the $U(1)$ flavor symmetry.

1.3 The AGT Duality

Previous study of $\mathcal{N} = 2$ dualities showed that there are strong relations between the complex structures of the punctured Riemann surface, C , and the four dimensional gauge theories obtained by compactifying and then partially twisting the six dimensional $\mathcal{N} = (2, 0)$ theory on C . The most important and simple relation is that the irreducible flavor groups of the four dimensional gauge theory can be identified as punctures on the Riemann surface. The sewing and pinching operations of the Riemann surface are executed through these punctures. Table 1.3 is a brief list from the gauge theory-Riemann surface dictionary.

Gauge theory	Riemann surface
coupling parameter space	$\mathcal{M}(C)$
S-duality group	$\pi_1(\mathcal{M}(C))$
gauge groups	tubes of C
flavor groups	punctures on C
diagonal gauging flavor groups	sewing or adding handles
ungauging	pinching of C

Table 1.3: The dictionary between gauge theories and Riemann surfaces. $\mathcal{M}(C)$ is the complex structure moduli space of C , which is a punctured Riemann surface. $\pi_1(\mathcal{M}(C))$ is the fundamental group of $\mathcal{M}(C)$.

If we do not know the AGT duality and we want to find a correspondence between a gauge theory in Table 1.3 and a two-dimensional theory on C , what we can learn from the above table? On the gauge theory side, the theories are solved since both the Seiberg-Witten curves [3] and the partition functions of the $\mathcal{N} = 2$ superconformal quiver linear theories are known [57]. On the Riemann surface side, we do not know any exact quantities that we can use to check the relation suggested in Table 1.3. However, if there is a two dimensional theory

which has some quantities that can be checked with the four dimensional theory, the theory must satisfy the S-duality group. Moreover through gauging and ungauging we can exchange the roles of flavor groups and gauge groups. This is a hint that if there is a two dimensional theory on the Riemann surface, the theory should treat flavor group parameters—the masses, and gauge group parameters—the Coulomb branch parameters, on the same footing. These observations suggest that the two dimensional theory may be a conformal field theory. It is still amazing that, [7] Alday, Gaiotto, Tachikawa found the four dimensional $\mathcal{N} = 2$ $SU(2)$ superconformal field theories have an exact duality with the two dimensional Liouville field theory. The flavor groups and gauge groups are identified as primary state insertions, $V_\alpha = e^{\alpha\phi}$, in Liouville theory. α is the momentum of the state and ϕ is the complex scala field appearing in the Lagrangian of the Liouville theory

$$S = \int d^2\sigma \sqrt{g} \left[\frac{1}{4\pi} (\partial\phi)^2 + \mu e^{2b\phi} + \frac{Q}{4\pi} R\phi \right], \quad (1.3.1)$$

where R is the curvature of the metric g , and $Q = b + 1/b$.

In [58], Gaiotto generalized the duality to asymptotic free theories and in [59], Wyllard generalized the duality to the case of $SU(N)$ gauge theories and Toda field theories. $SU(N)$ flavor groups and gauge groups are identified as ordinary primary states, $V_\alpha = e^{\alpha\cdot\phi}$, where α , ϕ are N dimensional vectors and the dot means inner product of vectors. The Lagrangian of the A_{N-1} Toda field theory is

$$S = \int d^2\sigma \sqrt{g} \left[\frac{1}{8\pi} \partial\phi \cdot \partial\phi + \mu \sum_{i=1}^{N-1} e^{b\phi \cdot e_i} + \frac{Q \cdot \phi}{4\pi} R \right], \quad (1.3.2)$$

where e_i are the simple roots of the A_{N-1} algebra and $Q = (b + 1/b)\rho$. The ρ in Q is the Weyl vector of A_{N-1} . Although the Liouville theory can be considered as the A_1 Toda theory, in the following the Toda theories are specific for the cases when $N > 2$.

Instanton partition functions are important objects in AGT duality. The Nekrasov instanton partition function for a superconformal linear quiver theory is

$$Z_{\text{full}}(q, a, m; \epsilon_i) = Z_{\text{classical}} Z_{1\text{-loop}} Z_{\text{instanton}}, \quad (1.3.3)$$

where ϵ_i , $i = 1, 2$ are the so-called Ω -deformation parameters. In the semiclassical limit, i.e., in the limit $\epsilon_1 = -\epsilon_2 = \hbar$ and $\hbar \rightarrow 0$, the prepotential defined as

$$F(q, a, m) = \lim_{\hbar \rightarrow 0} \hbar^2 \log Z_{\text{full}}(q, a, m; \hbar, -\hbar), \quad (1.3.4)$$

can be identified as the prepotential obtained from the corresponding Seiberg-Witten curve [5]. Please refer to Appendix A.1 for more detailed formulae of $Z_{\text{instanton}}$.

1.3.1 $SU(2)$ Quiver Theory

As we reviewed in Section 1.2, $SU(2)$ superconformal theories are quite special, since they have only one type of flavor groups and correspondingly only one type of punctures on the dual Riemann surface. The four important proposals of [7] are

- $Z_{\text{instanton}}$ can be identified with Liouville conformal blocks.
- $Z_{1\text{-loop}}$ can be identified with three point functions of Liouville theory.
- Square integral of partition functions can be identified with Liouville correlation functions.
- The square of Seiberg-Witten differential can be identified with the semi-classical limit of energy-momentum tensor of Liouville theory.

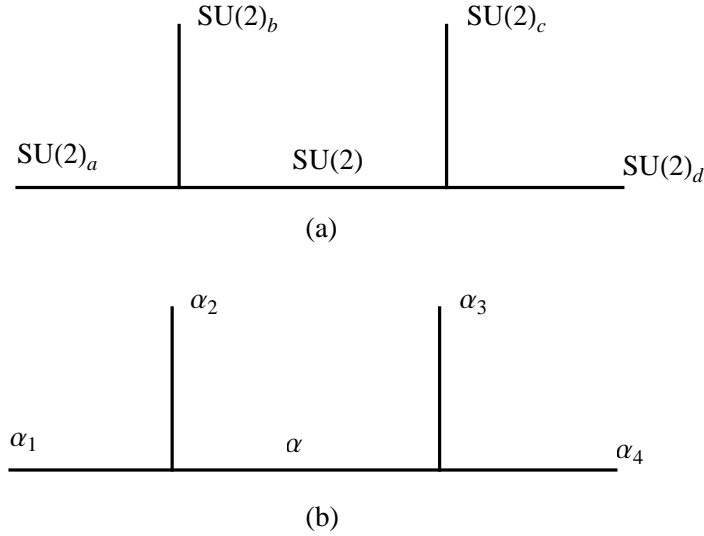


Figure 1.19: The prototype example of the AGT duality. (a) is the quiver diagram of the $SU(2)$, $N_f = 4$ theory. (b) is its dual Feynman diagram of the Liouville theory. The Greek letters are momenta of the primary states and they are also charges of the corresponding $SU(2)$ groups.

To realize these proposals there should be a dictionary between parameters of the two theories. In the following, we will give the dictionary and concentrate on the first proposal listed above. The prototype of the AGT duality (Figure 1.19) is the duality between the $SU(2)$

gauge theory with four flavors and the two dimensional Liouville theory with four primary states.

The instanton partition function of the gauge theory is the one-node form defined in Appendix A.1,

$$Z_{\text{inst}}^{\text{SU}(2), N_f=4} = \sum_{\vec{Y}} Z_{\text{fund}} Z_{\text{antifund}} Z_1, \quad (1.3.5)$$

with parameters $N = 1$, $a_i^{(1)} = a$, $a_2^{(1)} = -a$. The exact dictionary of parameters is given in Table 1.4 :

SU(2) Gauge theory	Liouville theory
Ω -deformation parameter (ϵ_1, ϵ_2)	the central charge $Q = b + \frac{1}{b}$ $b = \sqrt{\frac{\epsilon_1}{\epsilon_2}}$
Coulomb branch parameter a	internal momentum $\alpha = \frac{Q}{2} + \frac{a}{\sqrt{\epsilon_1 \epsilon_2}}$
four mass parameters (μ_1, μ_2) (μ_3, μ_4)	external momentum $\alpha_1 = \frac{Q}{2} + \frac{1}{2\sqrt{\epsilon_1 \epsilon_2}} (\mu_1 - \mu_2)$ $\alpha_2 = \frac{1}{2\sqrt{\epsilon_1 \epsilon_2}} (\mu_1 + \mu_2)$ $\alpha_4 = \frac{Q}{2} + \frac{1}{2\sqrt{\epsilon_1 \epsilon_2}} (\mu_3 - \mu_4)$ $\alpha_3 = \frac{1}{2\sqrt{\epsilon_1 \epsilon_2}} (\mu_3 + \mu_4)$

Table 1.4: The dictionary of parameters for the four flavor $SU(2)$ theory.

Notice that parameters in the first column of Table 1.4 are of mass dimension one, while parameters in the second column are massless. In order to identify the two sides, we have to divide parameters in the first column by the mass dimension one parameter $\sqrt{\epsilon_1 \epsilon_2}$. There is a $Q/2$ shift of a and the antisymmetric linear combinations of mass parameters. We can understand the shift in the following way. μ_1, μ_2 furnish a fundamental representation of $U(2)$, which can be decomposed into an $U(1) \subset SU(2)$ part plus an $U(1)$ part. Since the Coulomb branch parameter $(a, -a)$ is the charge of the $U(1)$, the Cartan subgroup of the $SU(2)$ gauge group, it is reasonable that a and the antisymmetric linear combinations of mass parameters are mapped in the same way.

The AGT duality tells us that

$$Z_{\text{inst}}^{\text{SU}(2), N_f=4}(a, \mu; \epsilon) = (1 - q)^{2\alpha_2(Q - \alpha_3)} \mathcal{F}(\alpha_1, \alpha_2, \alpha, \alpha_3, \alpha_4; q), \quad (1.3.6)$$

where $\mathcal{F}(\alpha_1, \alpha_2, \alpha, \alpha_3, \alpha_4; q)$ is the conformal block depicted in Figure 1.19. The four Liouville primary states are inserted at $\infty, 1, q, 0$ with momenta $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. The prefactor of the conformal block is named as the U(1) factor, which comes from the center of mass motion of the two D4-branes in Figure 1.7(a).

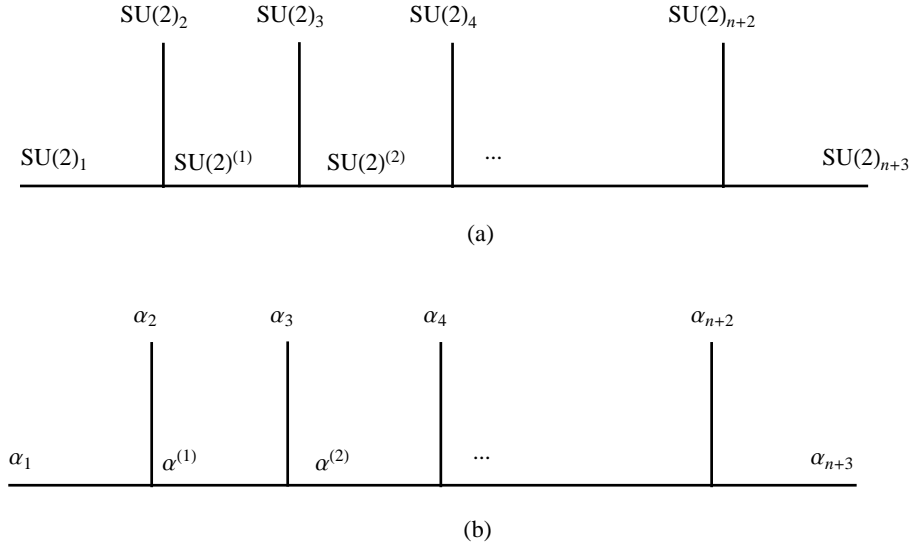


Figure 1.20: A general $SU(2)$ linear quiver theory and its Liouville theory dual.

With this remarkable example at hand, the generalization of the duality to a general n -node linear quiver theory can be inferred in the following way. In general the Liouville theory Feynman diagrams are the quiver diagrams without boxes and circles (Figure 1.20). The first two primary states correspond to two fundamental hypermultiplets and the last two primary states correspond to two antifundamental hypermultiplets. Intermediate primary states correspond to bifundamental hypermultiplets. Since bifundamental hypermultiplets have $U(1)$ flavor symmetries, the i -th bifundamental mass m_i is mapped directly to $\alpha_{i+2} = m_i / \sqrt{\epsilon_1 \epsilon_2}$. The Coulomb branch parameter at the i -th node is mapped to the i -th internal momentum with a $Q/2$ shift. Of course contributions from the nontrivial $U(1)$ factors should be included. The explicit formula is

$$Z_{\text{inst}}^{U(2)\text{linearquiver}}(q_i; a_i; \mu_i) = Z^{U(1)\text{linear}}(q_i; \mu_i) \mathcal{F}(\alpha_1, \alpha_2, \alpha^{(1)}, \alpha_3, \dots, \alpha_{n+1}, \alpha^{(n)}, \alpha_{n+2}, \alpha_{n+3}; z_i), \quad (1.3.7)$$

where

$$\begin{aligned}
\alpha_1 &= \frac{Q}{2} + \frac{1}{\sqrt{\epsilon_1 \epsilon_2}} (\mu_1 - \mu_2) , \\
\alpha_2 &= \frac{1}{\sqrt{\epsilon_1 \epsilon_2}} (\mu_1 - \mu_2) , \\
\alpha_{n+2} &= \frac{1}{\sqrt{\epsilon_1 \epsilon_2}} (\mu_3 + \mu_4) , \\
\alpha_{n+3} &= \frac{Q}{2} + \frac{1}{\sqrt{\epsilon_1 \epsilon_2}} (\mu_3 - \mu_4) ,
\end{aligned}$$

and for $i = 1, 2, \dots, n$,

$$\begin{aligned}
\alpha_{i+2} &= \frac{m_i}{\sqrt{\epsilon_1 \epsilon_2}} , \\
\alpha^{(i)} &= \frac{Q}{2} + \frac{1}{\sqrt{\epsilon_1 \epsilon_2}} a_i .
\end{aligned}$$

The $n+3$ primary states are inserted at z_i , which are related to the UV coupling constants in the following way, $z_1 = \infty$, $z_2 = 1$, $z_{i+2} = \prod_{j=1}^i q_j$, $z_{n+3} = 0$. The U(1) factor is

$$Z^{U(1)\text{linear}}(q_i; \mu_i) = \prod_{i \leq j} \left(1 - \prod_{k=i}^j q_k \right)^{2\alpha_{i+1}(Q - \alpha_{j+1})} ,$$

where the first product is over i, j for $i, j = 1, \dots, n$ and $i \leq j$.

For an n -node elliptic quiver theory which has n gauge nodes and n bifundamental hypermultiplets, the proposed dual CFT is the Liouville theory on a torus with n primary fields.

$$Z_{\text{inst}}^{U(2), \text{elliptic}}(q_i, a_i, m_i) = Z_{\text{elliptic}}^{U(1)} \mathcal{F}(\alpha_i; z_i) , \quad (1.3.8)$$

where α_i are internal or external momenta if i is odd or even respectively. The relations between parameters are

$$\begin{aligned}
\alpha_{2i+1} &= \frac{Q}{2} + \frac{a_i}{\sqrt{\epsilon_1 \epsilon_2}} , \\
\alpha_{2i} &= \frac{m_i}{\sqrt{\epsilon_1 \epsilon_2}} , \\
z_i &= \prod_{j=1}^{i-1} q_j .
\end{aligned}$$

The corresponding U(1) factor is

$$Z_{\text{elliptic}}^{U(1)} = \frac{\prod_{i=1}^n \prod_{k=0}^{\infty} \left(1 - \prod_{j=0}^k q_{i+j}\right)^{2\alpha_{2i-2}(Q-\alpha_{2i+2k})}}{\prod_{l=1}^{\infty} \left(1 - \prod_{i=1}^n q_i^l\right)}.$$

1.3.2 The $SU(N)$ Generalization

The $SU(N)$ generalization of the AGT duality is proposed in [59]. It was conjectured that when $N > 2$ the two dimensional CFT duals of the $SU(N)$ quiver gauge theories are the A_{N-1} Toda field theories. Although the Toda field theories are the natural generalizations of the Liouville theory, there is a big difference between them. In the Liouville theory, due to the power of Virasoro algebra, four point correlation functions are determined by three point functions. However, this is no longer true for the Toda field theories [60]. The reason is that the Toda field theories have more degrees of freedom than the Liouville theory, but their symmetry algebras, \mathcal{W} algebras, do not simplify the theories enough. We can expect that there is a subset of \mathcal{W} primary states whose higher point correlation functions can be reduced to three point ones. Indeed, it was found in [61, 62] that the expected subset exists and the momenta of primary states in this subset should satisfy

$$\alpha = \kappa\omega_1, \quad \text{or} \quad \alpha = \kappa\omega_{N-1}, \quad (1.3.9)$$

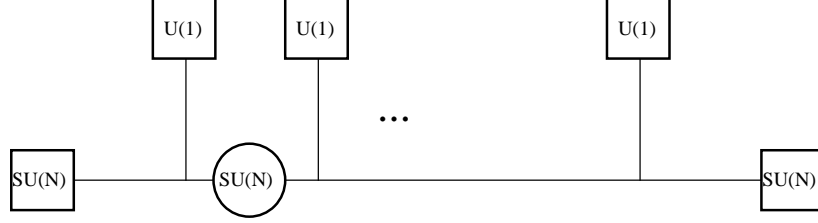
where κ is a complex constant, and ω_1 (ω_{N-1}) is the highest weight of the fundamental (antifundamental) representation of the A_{N-1} Lie algebra. The states in this subset is named as *semi-degenerate* states. It follows that the n -point correlation functions,

$$\langle \alpha_1 | V'_{\kappa_2} \cdots V'_{\kappa_{n-1}} | \alpha_n \rangle, \quad (1.3.10)$$

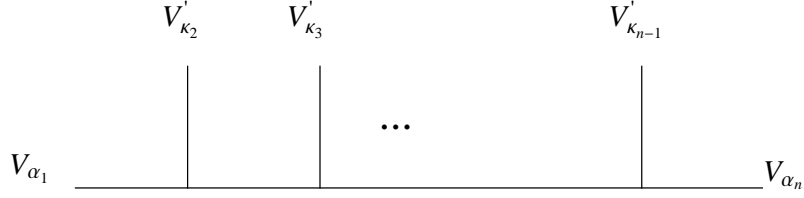
can be calculated in terms of \mathcal{W} conformal blocks and three-point functions of semi-degenerate states. In above formula, we use V'_{κ} to denote a semi-degenerate state whose momenta satisfy (1.3.9).

As reviewed in Section 1.2.3, when $N > 2$, an $SU(N)$ theory has multiple types of punctures and there are two special ones: full punctures and basic punctures. Considering the flavor symmetries, it is natural to propose that the full punctures should be identified with generic primary states, while the basic punctures should be identified with the semi-degenerate states [59]. The linear quiver theory and its Toda theory dual are shown in Figure 1.21. Once again, we see that if we strip off the nodes from the quiver diagram of a linear quiver gauge theory what we obtain is the Feynman diagram of the CFT dual of the quiver

gauge theory.



(a) $SU(N)$ linear quiver gauge theory



(b) A_{N-1} Toda conformal block

Figure 1.21: The AGT duality for $SU(N)$ theories. A $SU(N)$ linear quiver gauge theory is shown in (a) and its Toda theory duality is shown in (b).

In [59], the identification of the instanton partition function of $SU(3)$, $N_f = 6$ gauge theory with the conformal block of A_2 Toda theory of the form in (1.3.10) (where $n = 4$) is checked up to one instanton correction. The corresponding conformal block is shown in Figure 1.22. The results are enough to give the following dictionary of parameters,

$$\begin{aligned}\alpha_1 &= (\mu_1, \mu_2, \mu_3) + Q, \\ \kappa_2 &= \sum_{i=1}^3 \mu_i + 3Q, \\ \alpha &= (a_1, a_2, a_3) + Q, \\ \kappa_3 &= \sum_{i=4}^6 \mu_i + 6Q, \\ \alpha_4 &= (\mu_4, \mu_5, \mu_6) + Q,\end{aligned}$$

where (μ_1, μ_2, μ_3) is the vector of masses of fundamental hypermultiplets, (μ_4, μ_5, μ_6) is the vector of masses of antifundamental hypermultiplets and (a_1, a_2, a_3) the vector of Coulom-

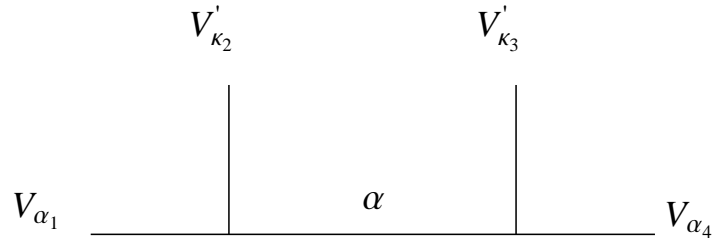


Figure 1.22: The conformal block checked for the A_2 Toda theory

b branch parameters. Notice that the A_2 Toda conformal weights α, α_1 and α_4 are three-dimensional vectors and κ_2, κ_3 are complex scalars in (1.3.9).

It was also proved in [59] that the perturbative part of $SU(N)$, $N_f = 2N$ partition function can be identified with the corresponding three point functions of semi-degenerate states.

Chapter 2

Polymorphism of Vortices

The issues we will discuss in this chapter have to do with the interplay between different incarnations of counting problems in gauge and string theory. More precisely, we will compute a given set of quantities which admit different interpretations depending on the point of view one takes. These different perspectives can be listed as follows:

- Classical limit of surface operators in $\mathcal{N} = 2$ four dimensional supersymmetric gauge theories.
- Supersymmetric index of the two dimensional gauge theory on the defect surface.
- Chern-Simons theory on a Lagrangian submanifold of the dual toric Calabi-Yau geometry.
- Partition functions for nonabelian vortices.
- AGT-dual as Toda conformal blocks with suitable degenerate field insertions.

The first perspective can be obtained via a D-brane construction by suspending N D4-branes between two parallel NS5-branes and then by extending N_f D2-branes between the D4-branes and an external parallel NS5'-brane (see Figure 2.1) [13]. By moving one of the initial NS5-brane to infinity, one freezes the gauge theory dynamics, letting the system at a classical phase [10].

The second point of view corresponds to focus on the leftover dynamics on the D2-branes [13]. Its vacua structure is characterized by vortex configurations whose partition function should be systematically computed. We make a detailed analysis of the derivation of these results from instanton counting and compare with the related studies by Nekrasov and Shatashvili [63].

The third corner is the viewpoint of the topological string on the system via geometric engineering. Indeed, the D2/D4/NS5 system can be recast as the topological vertex [38, 40] on the strip with suitable representations on the external legs [39].

Finally, the AGT dual of the four dimensional gauge theory computation is produced by representing the surface operators in the gauge theory [64] as degenerate fields insertions in the Toda A_{N-1} theory [8, 9, 10]. Related topics are also discussed in [65, 66, 67, 68, 69] from quantum field theory and integrable system point of view and in [70, 71, 72, 73] from matrix model, topological vertex and algebraic geometry point of view. As we will see, the insertion point coordinates get interpreted as open moduli or vortex counting parameters and the nonabelian vortex partition function can be interpreted as multiple interacting simple surface operators.

In the following we first compute the vortex partition functions for adjoint and antifundamental matter in supersymmetric $\mathcal{N} = (2, 2)$ gauge theories on the two dimensional plane via equivariant localization. Then we compute the topological vertex on the strip with boundary conditions corresponding to one-dimensional Young tableaux on a side and empty or transposed tableaux on the other and we show that the field theory limit of the open topological string amplitudes is equal to the vortex partition functions. After that, we reshuffle the field theory limit of the vertex partition functions in terms of generalized hypergeometric functions and therefore recover an AGT dual description in terms of degenerate Toda conformal blocks. A more exact AGT like duality for vortices will be discussed in next chapter, where the degeneration phenomenon of instanton partition functions relates the vortex partition functions to certain fusion channels of conformal blocks naturally. In the final section of this chapter, we discuss analogous re-summation formulae for the topological open string amplitudes in terms of q -deformed generalized hypergeometric functions.

2.1 Counting Vortices

In this section we analyze the moduli space of vortices for $U(N)$ gauge theories with an adjoint hypermultiplet, $N_f = N$ fundamental matter multiplets and $N_a = N$ multiplets in the antifundamental representation. The moduli space for $N_a = 0$ and without the adjoint hypermultiplet was analyzed in [13] via a proper IIA D-brane construction. This, as displayed in Figure 2.1, is obtained by considering a set of k parallel D2 branes of finite size in one dimension suspended between an NS5-brane and N (semi-)infinite D4-branes. The brane construction is also displayed in Table 2.1.

Interestingly, this moduli space was found to be a holomorphic submanifold of the moduli

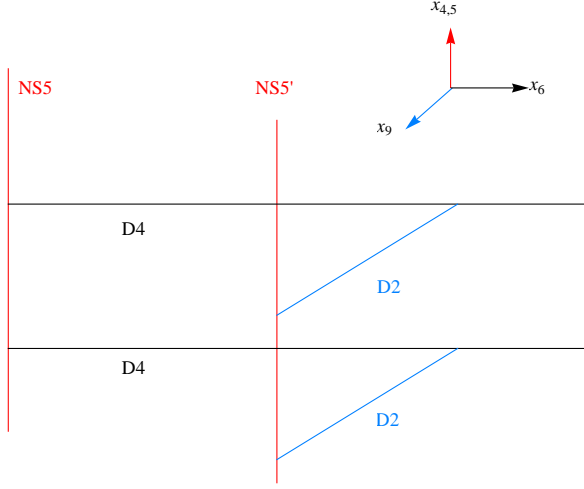


Figure 2.1: Brane construction of vortex moduli space

$brane \backslash dim$	0	1	2	3	4	5	6	7	8	9
NS5	○	○	○	○	○	○	×	×	×	×
NS5'	○	○	×	×	○	○	×	○	○	×
D4	○	○	○	○	×	×	○	×	×	×
D2	○	○	×	×	×	×	×	×	×	○

Table 2.1: The configuration of the IIA branes: ○ means parallel directions of branes; × means perpendicular directions of branes.

space of instantons for an $U(N)$ $\mathcal{N} = 2$ supersymmetric gauge theory in four dimensions. An ADHM-like construction of the vortex moduli space was carried out in [74] directly from field theory analysis and shown in [75] to be equivalent for $N_f = N$ to the D-brane construction of [13]. Here we will extend this analysis to the presence of adjoint and antifundamental matter and show that the relevant vortex moduli spaces can be obtained as holomorphic submanifolds of the instanton moduli space of four dimensional $\mathcal{N} = 2^*$ and $\mathcal{N} = 2$ $N_f = N$ $U(N)$ gauge theories respectively. Moreover, we will use equivariant localisation techniques to compute the relevant partition functions by vortex counting.

Let us first recall the $\mathcal{N} = 4$ ADHM construction following the notations of [76]. The ADHM data can be extracted from the low-energy dynamics of a system of N $D3$ -branes and k $D(-1)$ in flat space. In particular, the matrix model action for the k $D(-1)$ branes contains five complex fields $B_\ell, \phi \in \text{End}(V)$, $V = \mathbb{C}^k$ with $\ell = 1, \dots, 4$ in the adjoint representation of $U(k)$ describing the positions of the k $D(-1)$ -instantons in ten-dimensional space. In addition

open strings stretching between D(-1)-D3 branes provide two complex moduli I, J in the (\bar{k}, N) and (\bar{N}, k) bifundamental representations respectively of $U(k) \times U(N)$, that is $I \in \text{Hom}(W, V)$ and $J \in \text{Hom}(V, W)$ with $W = \mathbb{C}^N$. The ADHM constraints can be read as D and F-term equations of the matrix model action

$$\begin{aligned} [B_\ell, B_\ell^\dagger] + II^\dagger - J^\dagger J &= \zeta \ , \\ [B_1, B_2] + [B_3^\dagger, B_4^\dagger] + IJ &= 0 \ , \\ [B_1, B_3] - [B_2^\dagger, B_4^\dagger] &= 0 \ , \\ [B_1, B_4] + [B_2^\dagger, B_3^\dagger] &= 0 \ , \end{aligned} \tag{2.1.1}$$

together with

$$\begin{aligned} B_3 I - B_4^\dagger J^\dagger &= 0 \\ B_4 I + B_3^\dagger J^\dagger &= 0 \ . \end{aligned} \tag{2.1.2}$$

The $\mathcal{N} = 4$ instanton moduli space arises as a hyperkahler quotient with respect to a $U(k)$ group action with the above momentum maps (2.1.1) and (2.1.2). We can obtain the vortex moduli space for the $\mathcal{N} = (4, 4)$ theory in two dimensions by applying to the ADHM data (2.1.1), (2.1.2) the same procedure developed in [13], namely by considering the Killing vector field rotating the instantons in a plane and setting to zero the associated Hamiltonian. The vortices correspond then to instanton configurations which are invariant under the selected rotation group. To be explicit, let us consider the following $U(1)$ action on the ADHM data

$$\begin{aligned} (B_1, B_2, B_3, B_4) &\rightarrow (B_1, e^{i\theta} B_2, B_3, e^{-i\theta} B_4) \ , \\ (I, J) &\rightarrow (I, e^{i\theta} J) \ . \end{aligned} \tag{2.1.3}$$

This is a Hamiltonian action with generating vector field

$$\xi = \text{Tr} (B_2 \partial / \partial B_2 - B_4 \partial / \partial B_4 + J \partial / \partial J - h.c.) \ , \tag{2.1.4}$$

and Hamiltonian

$$H = \text{Tr} (B_2 B_2^\dagger + B_4 B_4^\dagger + J J^\dagger) \ . \tag{2.1.5}$$

Indeed we have

$$i_\xi \omega^{(1,1)} = dH \ , \tag{2.1.6}$$

with the Kahler form

$$\omega^{(1,1)} = dB_\ell \wedge dB_\ell^\dagger + dJ^\dagger \wedge dJ + dI \wedge dI^\dagger. \quad (2.1.7)$$

By restricting the $\mathcal{N} = 4$ ADHM data to the zero locus of the Hamiltonian (2.1.5) we get a holomorphic submanifold described by the data $(B_1, B_3 = \Phi)$ and I subject to the constraints

$$\begin{aligned} [B_1, B_1^\dagger] + [\Phi, \Phi^\dagger] + II^\dagger &= \zeta, \\ [B_1, \Phi] &= 0, \end{aligned} \quad (2.1.8)$$

together with the stability condition $\Phi I = 0$. The above data describe the moduli space of k vortices for $U(N)$ $\mathcal{N} = (4, 4)$ gauge theory in two dimensions as a Kahler quotient with $U(k)$ group action. Indeed, (2.1.8) are the D-term equations for a supersymmetric Euclidean D0-D2 system, whose lagrangian can be obtained from the reduction of the $\mathcal{N} = 2$ gauge theory in four dimension with $N_f = N$ fundamentals. Its bosonic part reads

$$\begin{aligned} \mathcal{L} = & \text{Tr} \left[\frac{1}{2} [\Phi, \Phi^\dagger]^2 + \frac{1}{2} \left([B_1, B_1^\dagger] + II^\dagger - \zeta \mathbf{1} \right)^2 + \{\Phi, \Phi^\dagger\} II^\dagger + |[B_1, \Phi]|^2 + |[B_1, \Phi^\dagger]|^2 \right. \\ & \left. + \frac{1}{2} [\varphi, \varphi^\dagger]^2 + |[\varphi, \Phi]|^2 + |[\varphi^\dagger, \Phi]|^2 + |[\varphi, B_1]|^2 + |[\varphi^\dagger, B_1]|^2 + \{\varphi, \varphi^\dagger\} II^\dagger \right], \end{aligned} \quad (2.1.9)$$

where Φ and φ are the two complex scalars coming from the reduction of the four dimensional vector field. The first line of (2.1.9), that is the φ independent part of the potential, can be rewritten as

$$\begin{aligned} & \text{Tr} \left[\frac{1}{2} [\Phi, \Phi^\dagger]^2 + \frac{1}{2} \left([B_1, B_1^\dagger] + II^\dagger - \zeta \mathbf{1} \right)^2 + \{\Phi, \Phi^\dagger\} II^\dagger + |[B_1, \Phi]|^2 + |[B_1, \Phi^\dagger]|^2 \right] = \\ & = \text{Tr} \left[\frac{1}{2} \left([B_1, B_1^\dagger] + [\Phi, \Phi^\dagger] + II^\dagger - \zeta \mathbf{1} \right)^2 + 2\Phi II^\dagger \Phi^\dagger + 2|[B_1, \Phi]|^2 \right], \end{aligned} \quad (2.1.10)$$

while the second line of (2.1.9) contains the equivariant action on the fields generated by φ .

The D-term equations of (2.1.10) correspond to the reduced $\mathcal{N} = 4$ ADHM equations (2.1.8). It can also be shown that the vortex action can be obtained from the $\mathcal{N} = 2^*$ action upon reduction under the Hamiltonian symplectomorphism generated by (2.1.4).

The vortex moduli space in the presence of additional N antifundamental matter multiplets can be obtained with the same method by extending the above construction with antifundamental hypermultiplets with masses m_f , $f = 1, \dots, N$, in the original four dimensional theory. These contributes by giving extra fermion zero modes λ_f with equivariant action $\varphi \cdot \lambda_f + m_f \lambda_f$. We will now apply localization formulae in order to compute the vortex

partition function.

2.1.1 Counting Formulae

In this subsection we perform the computation of the nonabelian vortex partition function via localization methods. Let's start with the case of the adjoint matter by computing the fixed points in the vortex moduli space under the torus action $T = T_{Cartan} \times T_h \times T_m$, where $T_{Cartan} = U(1)^N$ is the Cartan subgroup of the colour group¹, T_h is the lift to the vortices moduli space of the spatial rotation in \mathbb{R}^2

$$(B_1, \Phi, I) \rightarrow (e^{i\hbar} B_1, \Phi, I) , \quad (2.1.11)$$

and T_m the $U(1)_R$ symmetry

$$(B_1, \Phi, I) \rightarrow (B_1, e^{im} \Phi, I) , \quad (2.1.12)$$

where m is the mass parameter of the four dimensional adjoint hypermultiplet breaking $\mathcal{N} = 4$ to $\mathcal{N} = 2^*$.

The classification of the fixed points proceeds in a way very similar to the instanton case, except that now, since only the B_1 variable is involved, these are labeled by column diagrams $\{1^{k_l}\}$ only, where $l = 1 \dots, N$ and $\sum_l k_l = k$ is the total vorticity². In order to compute the determinants weighting the enumeration of fixed points in the localization formula, we evaluate the equivariant character on the tangent space around the fixed points which provides the relevant eigenvalues.

The total equivariant character can be computed to be

$$\tilde{\chi} = V^* \otimes V (T_h + T_m^{-1} - 1 - T_m^{-1} T_h) + V^* \otimes W (1 - T_m^{-1}) = (1 - T_m^{-1}) \chi , \quad (2.1.13)$$

where the reduced character χ is given by

$$\chi = V^* \otimes V (T_h - 1) + W^* \otimes V . \quad (2.1.14)$$

By exploiting the weight decomposition of the vector spaces

$$V = \sum_{l=1}^N \sum_{i=1}^{k_l} T_{a_l} T_h^{i-1}, \quad W = \sum_{l=1}^N T_{a_l}, \quad (2.1.15)$$

¹Notice that the colour group is identified with the flavour group in the two dimensional theory after ungauging and therefore the Cartan parameters become the mass parameters for the fundamental multiplets.

²See also the paper [19] for a similar computation.

one easily computes the reduced character to be

$$\chi = \sum_{l,m=1}^N \sum_{i=1}^{k_l} T_{a_{lm}} T_{\hbar}^{-k_m+i-1}. \quad (2.1.16)$$

From (2.1.13), (2.1.14) and (2.1.16) we get the determinant factor associated to a specific partition $\mathbf{k} = (k_1, \dots, k_N)$,

$$Z_{\mathbf{k}}^{adj} = \prod_{l,m} \prod_{i=1}^{k_l} \frac{a_{lm} + (-k_m + i - 1) \hbar - m}{a_{lm} + (-k_m + i - 1) \hbar}, \quad (2.1.17)$$

which is the partition function in presence of an adjoint multiplet of mass m . In the infinite mass limit this provides a derivation of the partition function corresponding to the $N_f = N$ theory

$$Z_{\mathbf{k}}^{vect} = \prod_{l,m} \prod_{i=1}^{k_l} \frac{1}{a_{lm} + (-k_m + i - 1) \hbar}. \quad (2.1.18)$$

Notice that the $m \rightarrow 0$ limit of (2.1.17) reduces to one. This is the expected result since in this limit we are recovering an enhanced $\mathcal{N} = (4, 4)$ supersymmetric theory, which therefore we prove to compute the Euler characteristic of the vortex moduli space.

Computing the partition function of vortices in presence of N antifundamentals with arbitrary masses amounts to shift the reduced character χ by a factor $\delta\chi = -T_m V$ (see [76]), where now $T_m = \otimes_{f=1}^N T_{m_f}$ is the generator of the $U(1)^{N_f}$ subgroup in $U(N_f)$. The direct computation then gives,

$$Z_{\mathbf{k}}^{af} = \frac{\prod_l \prod_f \prod_{i=1}^{k_l} a_l + (i - 1) \hbar + m_f}{\prod_{l,m} \prod_{i=1}^{k_l} a_{lm} + (-k_m + i - 1) \hbar}, \quad (2.1.19)$$

that coincides with the result obtained by different methods in [30], up to a shift $m_f \rightarrow m_f + \hbar$.

The generating functions for the abelian case are very simple, namely

$$\mathcal{Z}_{U(1)}^{vect} = \sum_{k=0}^{\infty} Z_{U(1),k}^{vect} z^k = \sum_{k=0}^{\infty} z^k \prod_{i=1}^k \frac{1}{i\hbar} = \exp\left(\frac{z}{\hbar}\right), \quad (2.1.20)$$

for the pure vector contribution, while in presence of adjoint and antifundamental one gets

respectively

$$\begin{aligned} Z_{U(1)}^{adj} &= \sum_{k=0}^{\infty} Z_{U(1),k}^{adj} z^k = \sum_{k=0}^{\infty} z^k \prod_{i=1}^k \frac{i + \frac{m}{\hbar}}{i} = (1 - z)^{-\frac{(m+\hbar)}{\hbar}}, \\ Z_{U(1)}^{af} &= \sum_{k=0}^{\infty} Z_{U(1),k}^{af} z^k = \sum_{k=0}^{\infty} z^k \prod_{i=1}^k \frac{\left(\frac{a+m}{\hbar} + i - 1\right)}{-i} = (1 + z)^{\frac{-(a+m)}{\hbar}}. \end{aligned} \quad (2.1.21)$$

These formulae match the results of [10].

2.1.2 Vortices from Instantons: The Moduli Spaces

It is worth to remark that the above vortex counting can be recovered directly from instanton counting by reducing to Young tableaux of column type and setting the sum of the two equivariant parameters to zero³.

Let us recall that the character of instanton counting in [79]

$$\chi_{inst} = \sum_{l,m} \sum_{s \in Y_l} T_{a_{lm}} \left(T_1^{-1_l(s)} T_2^{a_m(s)+1} + T_1^{1_l(s)+1} T_2^{-a_m(s)} \right), \quad (2.1.22)$$

where $a(s)$ and $1(s)$ are the “arm” and “leg” of the s^{th} box in the corresponding Young tableau. Restricting the above formula (2.1.22) to column diagrams, $Y_l = 1^{k_l}$, setting $T_1 T_2 = 1$ and denoting $T_2 = T_{\hbar}$, we get

$$\chi_{inst}^{red.} = \sum_{l,m} \sum_{i=1}^{k_l} T_{a_{lm}} \left[T_{\hbar}^{k_m(s)-i+1} + T_{\hbar}^{-k_m+i-1} \right] = \chi + \bar{\chi}, \quad (2.1.23)$$

where $\bar{\chi}$ is the vortex character (2.1.14) computed upon reflecting $\hbar \rightarrow -\hbar$.

Analogously, one can compute the fundamental and adjoint matter contributions. For the adjoint this is straightforwardly obtained by shifting by the mass m the formula for the vector multiplet, while the contribution to the instanton character of one (anti-)fundamental of mass m_f is [76]

$$\left[\delta \chi_{inst}^{af} \right]^{red.} = -T_{m_f} \sum_l \sum_{i=1}^{k_l} T_{a_l} T_{\hbar}^{i-1} = \delta \chi. \quad (2.1.24)$$

From the relation among the reduced instanton character and the vortex one one gets a straightforward relation among the associated partition functions. For example, for the case of

³An analogous reduction was considered in [77, 78] for the special partition $\{k_1, \dots, k_N\} = \{k, 0, \dots, 0\}$.

matter in the adjoint representation, one gets

$$\left[Z_{\mathbf{k}}^{inst}\right]^{red.}(\mathbf{a}, m, \hbar) = Z_{\mathbf{k}}^{adj}(\mathbf{a}, m, \hbar) Z_{\mathbf{k}}^{adj}(\mathbf{a}, -m, \hbar). \quad (2.1.25)$$

This alternative derivation, on the view of the A-model geometric engineering of Nekrasov partition function in [39], points to a relation with open topological string amplitudes on a strip where the reduction from arbitrary Young tableaux to columns is induced by suitably restricting the boundary conditions on the toric branes.

Analogous considerations, leading to the computation of two dimensional superpotentials via limits of the instanton partition function, were presented in [63]. We would like to underline that our scaling limit is different and that, as we will discuss at the beginning of next section, corresponds to a classical limit in four dimensional gauge theories. Indeed, the Nekrasov-Shatashvili limit corresponds to send $\epsilon_2 \rightarrow 0$ at fixed coupling, while, as shown in [80], the vortex partition functions can be recovered in a scaling limit in which also the gauge coupling is involved. This on one side confirms our interpretation of the vortex counting as a classical limit of the four dimensional gauge theory and moreover suggests that our result could represent a specific sector of the Nekrasov-Shatashvili's one.

2.2 Vortices from Vertices

In this section we will describe the topological open string counterpart of the vortex counting functions by using the topological vertex formalism. The conventions on the topological vertex are given in Appendix A.3. The vortex partition function is identified with the classical limit $\Lambda \rightarrow 0$ of the four dimensional gauge theory surface operator evaluation [10]. In the brane construction, this limit is realized by scaling to infinity the extension of the D4-brane in the x^6 direction (Figure 1.2). It is known that each IIA brane construction has a toric geometry engineering [37]. From the viewpoint of the toric geometry engineering of the four dimensional $\mathcal{N} = 2$ gauge theory, this limit corresponds to send to infinity the ladders of the relevant toric diagram, leaving us with a pure strip geometry, see Figure 2.2. It is interesting to notice that by a small deformation of the IIA brane diagram, we get the toric diagram.

As we will show in the following, the presence of the D2-branes is exactly taken into account by suitable boundary conditions on the topological vertex on the strip. In particular, in the case of antifundamental matter one has to place on the internal legs column diagrams with lengths k_l $l = 1, \dots, N$, corresponding to the number of D2-branes ending on the l -th D4 brane, see Figure 2.3(a). These correspond exactly to the column partitions of the total vorticity introduced in Section 2.1. The case of adjoint matter can be reproduced in the same

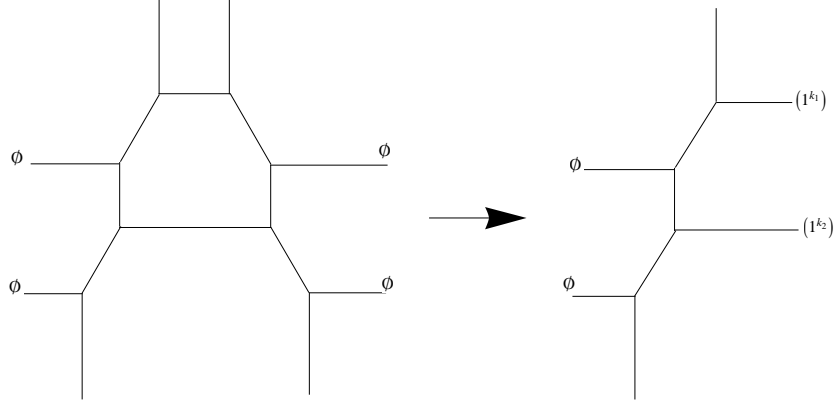


Figure 2.2: Toric diagram engineering the 4D gauge theory and its classical limit to the strip.

setup by identifying the boundary conditions on the horizontal direction of the toric diagram, see Figure 2.3(b). This identification comes from the periodicity of the D-brane construction engineering the $\mathcal{N} = 2^*$ theory.

2.2.1 Antifundamental Matters

In this subsection we compute the topological vertex on the strip with boundary conditions given by column Young tableaux of various lengths on one side of the strip and we show that there is a natural scaling limit on the Kahler moduli of the toric diagram amplitudes such that these reduce to the vortex counting partition functions with antifundamentals.

We start from the (normalized) topological vertex on a strip as calculated in [39]. Its form and some properties useful to our computations are given in the appendix.

Let us compute then topological vertex on the strip with boundary conditions corresponding to single columns representations on one side and trivial representations on the other. It reads

$$A_{\{\emptyset, \emptyset, \dots, \emptyset\}}^{\{1^{k_1}, 1^{k_2}, \dots, 1^{k_N}\}} = \prod_{l=1}^N \prod_{i=1}^{k_l} \frac{1}{1 - q^i} \frac{\prod_{l \leq m}^N \prod_{i=1}^{k_l} (1 - Q_{\alpha_l \beta_m} q^{(i-1)}) \prod_{l < m}^N \prod_{i=1}^{k_m} (1 - Q_{\beta_l \alpha_m} q^{-(i-1)})}{\prod_{l < m}^N \left(\prod_{i=1}^{k_m} (1 - Q_{\alpha_l \alpha_m} q^{i-1-k_l}) \prod_{i=1}^{k_l} (1 - Q_{\alpha_l \alpha_m} q^{1+k_m-i}) \right)}.$$

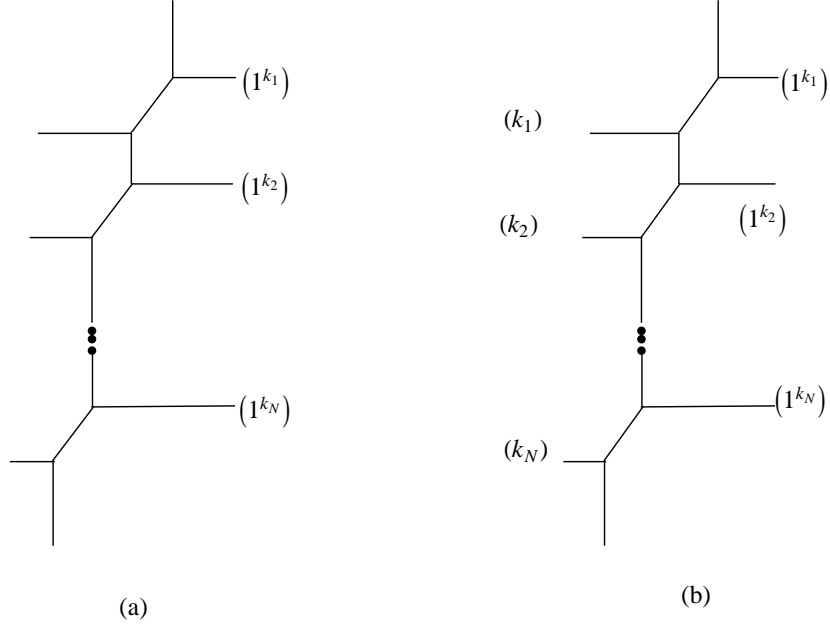


Figure 2.3: Strip diagrams: (a) antifundamental, (b) adjoint

By defining

$$\begin{aligned}
 Q_{\alpha_l \beta_f} &= e^{-\beta(a_l + m_f)} \quad (l \leq f), \\
 Q_{\beta_f \alpha_m} &= e^{\beta(a_l + m_f)} \quad (f < l), \\
 Q_{\alpha_l \alpha_m} &= e^{\beta a_{lm}}, \\
 q &= e^{-\beta \hbar},
 \end{aligned} \tag{2.2.1}$$

and going to the cohomological limit $\beta \rightarrow 0$ we find

$$A_{\{\emptyset, \emptyset, \dots, \emptyset\}}^{\{1^{k_1}, 1^{k_2}, \dots, 1^{k_N}\}} \rightarrow \prod_{l=1}^N \prod_{i=1}^{k_l} \frac{1}{i \hbar} \frac{\prod_{l \leq f}^N \prod_{i=1}^{k_l} (a_l + m_f + (i-1)\hbar) \prod_{f < l}^N \prod_{i=1}^{k_l} (a_l + m_f + (i-1)\hbar)}{\prod_{l < m}^N \prod_{i=1}^{k_m} (a_{lm} + \hbar(i-1-k_l)) \prod_{i=1}^{k_l} (a_{lm} + \hbar(i-1-k_m))},$$

which is easily recognized to be equal to (2.1.19).

2.2.2 Adjoint Matters

As we said, the adjoint matter case can be obtained by computing the topological vertex on the strip diagram of Figure 2.3.

The topological vertex computation gives, by using the properties listed in the Appendix,

$$\begin{aligned}
A_{\{k_1, k_2, \dots, k_N\}}^{\{1^{k_1}, 1^{k_2}, \dots, 1^{k_N}\}} &= \prod_{l=1}^N q^{k_l(k_l-1)/2} \prod_{i=1}^{k_l} \frac{1}{(1-q^i)^2} \prod_{i=1}^{k_l} (1-q^i Q_{\alpha_l \beta_l}) (1-q^{-i} Q_{\alpha_l \beta_l}) \times \\
&\quad \frac{\prod_{l < m} \prod_{i=1}^{k_l} (1-q^{i-1-k_m} Q_{\alpha_l \beta_m}) (1-q^{i-1-k_m} Q_{\beta_l \alpha_m})}{\prod_{l < m} \prod_{i=1}^{k_l} (1-q^{i-1-k_m} Q_{\alpha_l \alpha_m}) (1-q^{i-1-k_m} Q_{\beta_l \beta_m})} \times \\
&\quad \frac{\prod_{i=1}^{k_m} (1-q^{-i+1+k_l} Q_{\alpha_l \beta_m}) (1-q^{-i+1+k_l} Q_{\beta_l \alpha_m})}{\prod_{i=1}^{k_m} (1-q^{-i+1+k_l} Q_{\alpha_l \alpha_m}) (1-q^{-i+1+k_l} Q_{\beta_l \beta_m})}, \tag{2.2.2}
\end{aligned}$$

where $\alpha_l = (1^{k_l})$ and $\beta_l = \alpha_l^t = (k_l)$.

Via the identifications

$$\begin{aligned}
q &= e^{-\beta \hbar}, \\
Q_{\alpha_l \beta_l} &= e^{-\beta m}, \\
\text{and for } l &< m, \\
Q_{\beta_l \alpha_m} &= e^{-\beta(m+a_{lm})}, \\
Q_{\alpha_l \alpha_m} &= e^{-\beta a_{lm}}, \\
Q_{\beta_l \beta_m} &= e^{-\beta a_{lm}}, \\
Q_{\alpha_l \beta_m} &= e^{-\beta(a_{lm}-m)},
\end{aligned}$$

and by taking the $\beta \rightarrow 0$ limit, (2.2.2) reduces to

$$\begin{aligned}
&\prod_{l=1}^N \prod_{i=1}^{k_l} \frac{(i\hbar + m)(i\hbar - m)}{(\hbar i)^2} \prod_{l < m}^N \frac{\prod_{i=1}^{k_l} ((i-1-k_m)\hbar + a_{lm} - m)((i-1-k_m)\hbar + a_{lm} + m)}{\prod_{i=1}^{k_l} ((i-1-k_m)\hbar + a_{lm})((i-1-k_m)\hbar + a_{lm})} \\
&\times \prod_{l < m}^N \frac{\prod_{i=1}^{k_m} ((-i+1+k_l)\hbar + a_{lm} - m)((-i+1+k_l)\hbar + a_{lm} + m)}{\prod_{i=1}^{k_m} ((-i+1+k_l)\hbar + a_{lm})((-i+1+k_l)\hbar + a_{lm})}, \tag{2.2.3}
\end{aligned}$$

which equals to

$$Z_{\mathbf{k}}^{\text{adj}}(\mathbf{a}, m) Z_{\mathbf{k}}^{\text{adj}}(\mathbf{a}, -m).$$

2.3 Surface Operators and the Toda CFTs

In this section we discuss the re-summation formulae for supersymmetric vortex partition functions and interpret them in terms of suitable conformal blocks of Toda field theory. In particular we provide a closed expression for the generating functions of vortices in terms

of generalized hypergeometric functions, which in turn are the building blocks for amplitudes with degenerate field insertions in Toda conformal field theory. As anticipated in the introduction the origin of this relation has to be understood in terms of surface operators in four-dimensional $\mathcal{N} = 2$ superconformal gauge theory, namely they can be described in terms of a two dimensional gauge theory living on the defects where the surface operators lies.

In order to better clarify this issue, let us consider the brane realization of surface operators in $\mathcal{N} = 2$ SYM with $U(N)$ gauge group, see Figure 2.2. The gauge theory is realized as a set of N parallel D4-branes suspended between two parallel NS5 branes. The transverse distance between these two NS5-branes is proportional to $\ln \Lambda$, Λ being the dynamical scale of the gauge theory [3]. The surface operator is obtained by suspending k D2-branes between a further parallel and transversally displaced NS5'-brane and the D4-branes. The transverse distance is the dynamical scale of a two dimensional theory, namely its Fayet-Iliopoulos parameter. The location of the k D2-branes on the D4-branes determines a partition of $N = \sum_{a=1}^k N_a$ corresponding to the generically unbroken gauge symmetry $\prod_a U(N_a)$. We will consider the case of surface operators breaking to $U(1)^N$.

It was shown in [10] that the abelian vortex partition function computes the classical limit of simple surface operators. In this section we argue that the nonabelian vortex counting of the previous sections corresponds to the classical limit of interacting multiple surface operators of simple type in interaction. Restricting to the computation of the classical value of the above surface operators corresponds to move far away the two NS5-branes, therefore leaving the corresponding $U(N)$ theory non dynamical. So doing we are generalizing the brane realization of the vortices proposed in [13]. In particular the four dimensional gauge group becomes the flavor symmetry of the two dimensional gauge theory.

The gauge theory point of view also suggests looking for an AGT dual of the vortex partition function. Actually, having realized the vortex partition function in terms of the dual topological string as the vertex on the strip with single columns Young tableaux, we can formulate the Toda field theory dual along the lines elaborated in [9], that is by realizing the surface operator insertions as particular toric branes on the strip.

The AGT dual of the Nekrasov partition function of the $U(N)$ gauge theory with $2N$ fundamentals can be obtained by the Toda conformal block on the sphere with two maximal punctures, at 0 and ∞ , and two semi-degenerate fields at 1 and z . In this framework the dual of surface operators is realized by inserting further degenerate fields [8] in the Toda field theory conformal block. Indeed we are about to prove that the resummed vortex partition function can be expressed precisely in terms of these conformal blocks.

Let us focus on the case of antifundamental matter and consider the following generating

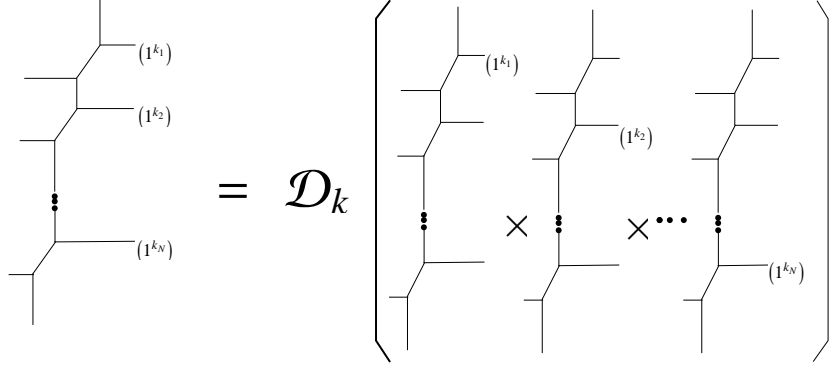


Figure 2.4: The strip amplitude for matter in the antifundamental

function

$$\mathcal{Z}^{af}(\mathbf{z}, m_f, a_l, \hbar) = \sum_{\mathbf{k}} \mathbf{z}^{\mathbf{k}} Z_{\mathbf{k}}^{af}, \quad (2.3.1)$$

where $\mathbf{k} = \{k_1, \dots, k_N\}$, $\mathbf{z} = \{z_1, \dots, z_N\}$ and $\mathbf{z}^{\mathbf{k}} = \prod_l z_l^{k_l}$. By making use of the identity

$$(a-l)_m (-a-m)_l = \left(1 + \frac{(m-l)}{a}\right)^{-1} (a+1)_m (-a+1)_l, \quad (2.3.2)$$

where $(a)_n = \prod_{i=1}^n (a+i-1)$ is the usual Pochhammer symbol we can rewrite the vortex partition function as

$$Z_{\mathbf{k}}^{af} = \prod_{l < m}^N \left(1 + \hbar \frac{(k_m - k_l)}{a_{ml}}\right) \prod_{l=1}^N \frac{1}{k_l!} \prod_f^N \left(\frac{a_l + m_f}{\hbar}\right)_{k_l} \left(\prod_{l \neq m}^N \left(\frac{a_{lm} + \hbar}{\hbar}\right)_{k_l}\right)^{-1}. \quad (2.3.3)$$

By replacing the latter in the definition (2.3.1), we then get

$$\mathcal{Z}^{af}(\mathbf{z}, m_f, a_l, \hbar) = D \prod_{l=1}^N {}_N F_{N-1}(A_l, B_l, z_l), \quad (2.3.4)$$

where ${}_N F_{N-1}(A, B, z) = \sum_{\mathbf{k}} \frac{z^{\mathbf{k}}}{k!} \frac{(A_1)_{k_1} \dots (A_N)_{k_N}}{(B_1)_{k_1} \dots (B_{N-1})_{k_{N-1}}}$ is the generalized hypergeometric function

and

$$\begin{aligned}
D &= \prod_{l < m}^N \left(1 + \hbar \frac{z_m \partial_{z_m} - z_l \partial_{z_l}}{a_{ml}} \right), \\
A_l &= \left\{ \frac{a_l + m_1}{\hbar}, \frac{a_l + m_2}{\hbar}, \dots, \frac{a_l + m_N}{\hbar} \right\}, \\
B_l &= \left\{ \frac{a_{l1} + \hbar}{\hbar}, \frac{a_{l2} + \hbar}{\hbar}, \dots, \frac{a_{lN} + \hbar}{\hbar} \right\}.
\end{aligned} \tag{2.3.5}$$

The AGT dual picture is then recovered by noticing that the generalized hypergeometric functions are the degenerate conformal blocks in Toda field theory considered in [62], namely the ones associated to the four point function

$$< \alpha_2 | V_{-b\omega_1}(z) V_{-\kappa\omega_{N-1}}(1) | \alpha_1 >, \tag{2.3.6}$$

where $|\alpha_1 >$ and $|\alpha_2 >$ are two primary states, $V_{-b\omega_1}$ is the highest weight degenerate field and $V_{-\kappa\omega_{N-1}}$ the vertex with momentum proportional to the lowest root. Each of them corresponds to the field theory limit of a single toric brane amplitude [9]. The total amplitude (2.3.4) is given by the action of the differential operator D in (2.3.5) over a product of N single brane amplitudes (see fig. 4). The nonabelian structure of the amplitude is encoded in the operator D of which it would be nice to provide a precise CFT transliteration.

As we have shown in Section 2.2, the vortex counting can be obtained from instanton counting by restricting to columns. This should have a clean counterpart in the AGT dual picture. Notice that the full amplitude is expressed in terms of correlators with a single degenerate field insertion. Therefore it should be possible to interpret (2.3.4) as a correlator on a degenerate sphere, with further insertions of degenerate fields on the stretching collars. In this way, the intermediate states would reduce to a tower of degenerate states which depend on the level only and thus could be represented as columns with height corresponding to the level.

Let us notice that the operator $z\partial_z$ acting on generalized hypergeometric functions produces linear combinations of them with shifted parameters. Therefore formula (2.3.4) can also be written in terms of products of linear combinations of generalized hypergeometric functions with shifted parameters.

Indeed, it is easy to uplift the previous procedure to the full open topological string ampli-

tude on the strip

$$A_{\{\emptyset, \emptyset, \dots, \emptyset\}}^{\{1^{k_1}, 1^{k_2}, \dots, 1^{k_N}\}} = \prod_{l < m}^N \frac{1 - Q_{\alpha_l \alpha_m} q^{k_l - k_m}}{1 - Q_{\alpha_a \alpha_b}} \left(\frac{Q_{\beta_l \alpha_m} q}{Q_{\alpha_l \alpha_m}} \right)_{k_m} \quad (2.3.7)$$

$$\prod_{l=1}^N \prod_{i=1}^{k_l} \frac{1 - Q_{\alpha_l \beta_l} q^{i-1}}{1 - q^i} \prod_{l < m}^N \prod_{i=1}^{k_l} \frac{1 - Q_{\alpha_l \beta_m} q^{i-1}}{1 - Q_{\alpha_l \alpha_m} q^i} \prod_{i=1}^{k_m} \frac{1 - Q_{\beta_l \alpha_m} q^{i-1}}{1 - Q_{\alpha_l \alpha_m} q^i}.$$

For $l < m$, we define $M_{l,m} = Q_{\alpha_l \beta_m} q^{-1}$; $M_{m,l} = Q_{\beta_l \alpha_m} q^{-1}$; $Q_{l,m} = Q_{\alpha_l \alpha_m}$; $Q_{m,l} = Q_{\beta_l \alpha_m} q^{-1}$, while for $l = m$, $M_{l,l} = Q_{\alpha_l \beta_l} q^{-1}$; $Q_{l,l} = 1$. By also defining

$$[Q]_k = \prod_{i=1}^k (1 - Q q^i), \quad (2.3.8)$$

$$\mathcal{D}_k = \prod_{l < m}^N \frac{1 - Q_{l,m} q^{k_l - k_m}}{1 - Q_{l,m}} \left(\frac{M_{m,l}^{-1}}{Q_{l,m}} \right)_{k_m},$$

we get

$$A_{\{\emptyset, \emptyset, \dots, \emptyset\}}^{\{1^{k_1}, 1^{k_2}, \dots, 1^{k_N}\}} = \prod_{m=1}^N \mathcal{D}_k \frac{\prod_{l=1}^N [M_{l,m}]_{k_l}}{[1]_{k_m} \prod_{n \neq m}^N [Q_{n,m}]_{k_n}}. \quad (2.3.9)$$

This is schematically encoded in Figure 2.4. By resumming the topological string amplitudes as

$$\mathcal{A}(\mathbf{z}) = \sum_{\mathbf{k}} \mathbf{z}^{\mathbf{k}} A_{\{\emptyset, \emptyset, \dots, \emptyset\}}^{\{1^{k_1}, 1^{k_2}, \dots, 1^{k_N}\}}. \quad (2.3.10)$$

we obtain

$$\mathcal{A}(\mathbf{z}) = \mathcal{D} \prod_{l=1}^N \Phi_{N-1}(X_l, Y_l, z_l), \quad (2.3.11)$$

where ${}_N\Phi_{N-1}(X, Y, z) = \sum_k \frac{z^k}{[1]_k} \frac{[X_1]_k \dots [X_N]_k}{[Y_1]_k \dots [Y_{N-1}]_k}$ is a q-deformed generalized hypergeometric function, $X_l = e^{-\beta \hbar(A_l - 1)}$, $Y_l = e^{-\beta \hbar B_l}$ and $\mathcal{D} = \prod_{l < m} \frac{1 - Q_{l,m} q^{z_l \partial_{z_l} - z_m \partial_{z_m}}}{1 - Q_{l,m}}$ up to a multiplicative redefinition of the open moduli \mathbf{z} . The operator \mathcal{D} is a finite difference operator whose action on the q-deformed generalized hypergeometric functions multiplicatively shifts their arguments. This result could be interpreted in the light of a five dimensional uplift of the AGT relation [81].

Let us now discuss the vortex partition function for the adjoint matter case. By making

use of the previous identity (2.3.2) we obtain

$$Z_{\mathbf{k}}^{adj} = \prod_{l < m} \frac{\left(1 - \hbar \frac{k_l - k_m}{a_{lm}}\right)}{\left(1 - \hbar \frac{k_l - k_m}{a_{lm} - m}\right)} \prod_l \frac{(m/\hbar + 1)_{k_l}}{k_l!} \prod_{l \neq m} \frac{\left(\frac{a_{lm} - m}{\hbar} + 1\right)_{k_l}}{\left(\frac{a_{lm}}{\hbar} + 1\right)_{k_l}} \prod_{l < m} \frac{\left(-\frac{a_{lm} + m}{\hbar} - k_l\right)_{k_m}}{\left(-\frac{a_{lm} - m}{\hbar} - k_l\right)_{k_m}}. \quad (2.3.12)$$

Notice that this form does not show an obvious resummation in terms of generalized hypergeometric functions due to the last multiplicative factor in (2.3.12). However, the open topological string amplitude in the $\beta \rightarrow 0$ limit (2.2.3) can be recast, by making use of (2.3.12), in the form⁴

$$\prod_{l < m} \frac{\left(1 - \hbar \frac{k_l - k_m}{a_{lm}}\right)^2}{\left(1 - \hbar \frac{k_l - k_m}{a_{lm} - m}\right) \left(1 - \hbar \frac{k_l - k_m}{a_{lm} + m}\right)} \prod_{l, m} \frac{\left(\frac{a_{lm} - m}{\hbar} + 1\right)_{k_l} \left(\frac{a_{lm} + m}{\hbar} + 1\right)_{k_l}}{\left(\left(\frac{a_{lm}}{\hbar} + 1\right)_{k_l}\right)^2}. \quad (2.3.13)$$

By re-summing the above coefficients against $\mathbf{z}^{\mathbf{k}}$ one finally gets

$$\mathcal{D}^{adj}(\mathbf{a}, m) \prod_l {}_{2N}F_{2N-1} \left(A_l^{adj}, B_l^{adj}, z_l \right), \quad (2.3.14)$$

where

$$\begin{aligned} A_l^{adj} &= \left(\frac{a_{lm} + m}{\hbar} + 1, \frac{a_{lm} - m}{\hbar} + 1 \right), \\ B_l^{adj} &= \left(\frac{a_{lm}}{\hbar} + 1, \frac{a_{lm}}{\hbar} + 1 \right), \end{aligned} \quad (2.3.15)$$

and

$$\mathcal{D}^{adj}(\mathbf{a}, m) = \prod_{l < m} \frac{\left(1 - \hbar \frac{z_l \partial_{z_l} - z_m \partial_{z_m}}{a_{lm}}\right)^2}{\left(1 - \hbar \frac{z_l \partial_{z_l} - z_m \partial_{z_m}}{a_{lm} - m}\right) \left(1 - \hbar \frac{z_l \partial_{z_l} - z_m \partial_{z_m}}{a_{lm} + m}\right)}. \quad (2.3.16)$$

The re-summed form (2.3.14) in terms of generalized hypergeometric functions suggests an interpretation of the re-summed open topological string amplitude in the $\beta \rightarrow 0$ limit as degenerate conformal blocks of Toda field theory on the sphere. We argue that, by using a suitable generalization of the results in [82] to Toda field theory, this can be recast as conformal blocks on the torus giving the expected AGT dual description.

As it is well known generalized hypergeometric functions satisfy generalized hypergeometric *differential* equations. Moreover, the q -deformed generalized hypergeometric functions, re-summing the vertex amplitudes, satisfy corresponding *finite difference* equations.

⁴Notice that in the product the two multiplicative unfair terms cancel.

Chapter 3

Degeneration of Instantons and the Exact CFT Duals of Vortices

As discussed in the previous chapter, nonabelian vortices can be interpreted as multiple insertion of degenerate fields. It is important to notice that in the presence of multiple insertions, the conformal blocks span a vector space whose dimension is fixed by the fusion rules of degenerate fields. In this chapter we will provide a full realization of the fusion rules of CFTs in terms of topological string amplitudes with general boundary conditions.

As a byproduct we identify the nonabelian vortex partition function with a specific fusion channel of the degenerate conformal block, different from the one considered so far in the literature.

Moreover, we realize the vortex counting problem as sub-counting instantons by showing how to relate the Nekrasov partition function and its vortex counterpart by a particular choice of mass parameters in an appropriately engineered gauge theory in four dimensions. On the gauge theory side, this boils down to consider surface operator insertions in a theory with a simpler quiver structure. On the AGT dual side, we notice that the above mass parameter assignments produce the insertion of degenerate fields in the Liouville/Toda CFT amplitudes. We study this correspondence in depth, reproduce some known results and embed them in a wider framework. In particular we show the correspondence between the fusion channel choice in the Liouville/Toda field theory side and the choice of possible surface operator insertions. The relation with topological strings, in the form of related strip amplitudes [38, 39], is also considered in full generality for the $SU(2)$ case and in some particular exemplificative ones for $SU(N)$.

We will first calculate the CFT dual of $SU(2)$ vortex partition functions. Then we extend the CFT dual for $SU(N)$ vortices and argue its validity for general strip amplitudes.

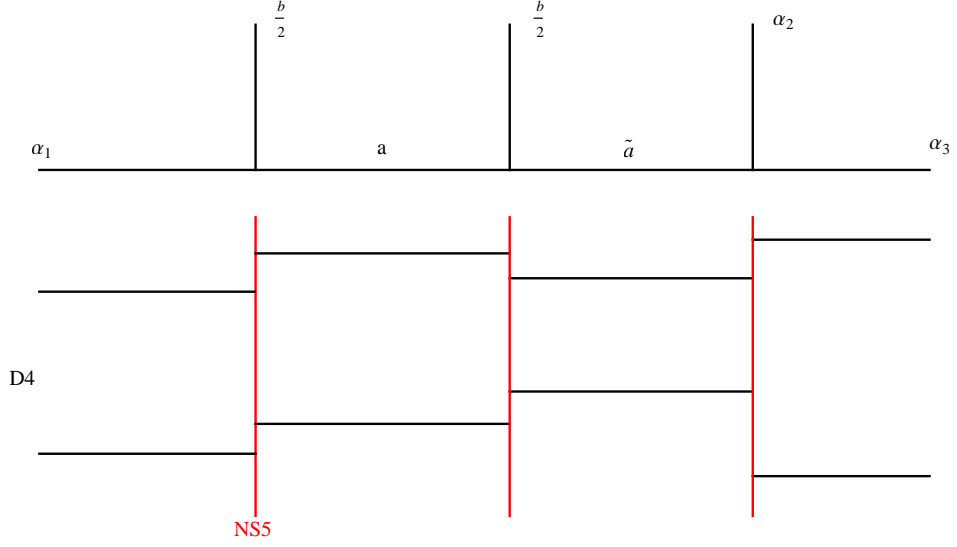


Figure 3.1: AGT relation between $SU(2)$ quiver gauge theory and CFT

3.1 $SU(2)$ Vortices and Degenerate States

3.1.1 General Setup

We start from the two-node $SU(2)$ theory with specific parameters. Its Liouville conformal block dual and brane construction is illustrated in 3.1. Following the results of previous chapter, we will focus on the free field limit, $\epsilon_+ := \epsilon_1 + \epsilon_2 = 0$. The parameters of this two-node quiver are: μ_1, μ_2 are masses of antifundamental hypermultiples; μ_3, μ_4 are masses of fundamental hypermultiples; m is the mass of bifundamental hypermultiplet and $(a_1, a_2) = (a, -a)$; $(\tilde{a}_1, \tilde{a}_2) = (\tilde{a}, -\tilde{a})$ are Coulomb branch parameters of the first and second gauge factor.

On the CFT side $\alpha_1, \alpha_2, \alpha_3$ are the external momenta in Liouville theory. When all parameters are generic, what we get is just the standard AGT correspondence between instanton partition functions of quiver gauge theories and conformal blocks with five operator insertions. When there are degenerate states, different fusion channels will give different results which also have different gauge theory interpretation as we will show in the following. For two node $SU(2)$ quiver theories there are two channels, one corresponding to $SU(2)$ vortex partition functions while the other to a simple surface operator as discussed in [9]. The general situation with the insertion of more degenerate fields is discussed in subsequent sections.

The standard AGT-relation, as reviewed in chapter one gives the following map between

parameters:

$$\begin{aligned}
\mu_1 &= \alpha_1 - \frac{\epsilon_2}{2}, \\
\mu_2 &= -\alpha_1 - \frac{\epsilon_2}{2}, \\
\mu_3 &= \alpha_2 + \alpha_3, \\
\mu_4 &= \alpha_2 - \alpha_3, \\
m &= -\frac{\epsilon_2}{2}.
\end{aligned} \tag{3.1.1}$$

The fusion rules of the Liouville field theory imply that

$$\begin{aligned}
\alpha_1 &= a - s_1 \frac{\epsilon_2}{2}, \\
\tilde{a} &= a - s_2 \frac{\epsilon_2}{2},
\end{aligned} \tag{3.1.2}$$

where s_1 and s_2 are ± 1 . This fixes the masses to be

$$\begin{aligned}
\mu_1 &= a - (s_1 + 1) \frac{\epsilon_2}{2}, \\
\mu_2 &= -a + (s_1 - 1) \frac{\epsilon_2}{2}.
\end{aligned}$$

Let us remark that when the differences between Coulomb branch parameters and fundamental/bifundamental masses are linear in ϵ_1 and ϵ_2 the instanton partition function is largely simplified. To see this let us recall the contribution from antifundamental fields

$$Z_{\text{antifund}}(m, a, Y) = \prod_{\alpha=1}^2 \prod_{(i,j) \in Y_\alpha} (a_\alpha + m + \epsilon_2(j - i)), \tag{3.1.3}$$

where $a_1 = a$; $a_2 = -a$, and (i, j) are the box location in the Young tableaux. If we choose $s_1 = -1$, then $\mu_1 = a$; $\mu_2 = -a - \epsilon_2$. The above formula then implies that $Y_2 = \emptyset$ and Y_1 to be a row. The other choice $s_1 = 1$ just exchanges the roles of Y_1 and Y_2 . So the choice of fusion channel here is just a convention. What is really relevant is the choice of s_2 . Notice that bifundamental masses can transfer degeneration between adjacent gauge factors of a quiver

theory. Indeed the contribution of bifundamental hypermultiples is

$$Z_{\text{bifund}}(m, a, \tilde{a}, Y, W) = \prod_{\alpha=1}^2 \prod_{\beta=1}^2 Z_{\text{bifund}}^{(\alpha, \beta)}, \quad (3.1.4)$$

$$Z_{\text{bifund}}^{(\alpha, \beta)} = \prod_{s \in Y_\alpha} \prod_{t \in W_\beta} (m_{\alpha, \beta} + \epsilon_2 (A_{Y_\alpha}(s) + L_{W_\beta}(s) + 1)) (m_{\alpha, \beta} - \epsilon_2 (A_{W_\beta}(t) + L_{Y_\alpha}(t) + 1)) ,$$

$$m_{\alpha, \beta} := a_\alpha - \tilde{a}_\beta - m .$$

From the second fusion relation in the diagram one gets

$$\begin{aligned} m_{1,1} &= (s_2 + 1) \frac{\epsilon_2}{2} , \\ m_{2,2} &= (1 - s_2) \frac{\epsilon_2}{2} , \\ m_{1,2} &= 2a + (1 - s_2) \frac{\epsilon_2}{2} , \\ m_{2,1} &= -2a + (1 + s_2) \frac{\epsilon_2}{2} , \end{aligned} \quad (3.1.5)$$

Moreover, the AGT duality implies that, up to a $U(1)$ factor which doesn't play any role here,

$$\begin{aligned} Z_{\text{Quiver}} \left(a, \tilde{a} = a - s_2 \frac{\epsilon_2}{2}; \mu_1 = a, \mu_2 = -a - \epsilon_2; \alpha_2 + \alpha_3, \alpha_2 - \alpha_3 \right) \\ = \mathcal{F} \left(a + \frac{\epsilon_2}{2}, \frac{-\epsilon_2}{2}, a, \frac{-\epsilon_2}{2}, a - s_2 \frac{\epsilon_2}{2}, \alpha_2, \alpha_3 \right) , \end{aligned} \quad (3.1.6)$$

where the LHS is the instanton partition function of $SU(2)$ quiver gauge theory and the RHS is the conformal block of Liouville field theory.

In the following we will show that when $s_2 = -1$ the quiver partition function in the above formula reduces to the $SU(2)$ vortex partition function, while when $s_2 = 1$, it corresponds to the $SU(2)$ simple surface operator.

3.1.2 $SU(2)$ Vortices

Let us start investigating the case $s_2 = -1$ where

$$\begin{aligned} m_{1,1} &= 0 , \\ m_{2,2} &= \epsilon_2 , \\ m_{1,2} &= 2a + \epsilon_2 , \\ m_{2,1} &= -2a . \end{aligned} \quad (3.1.7)$$

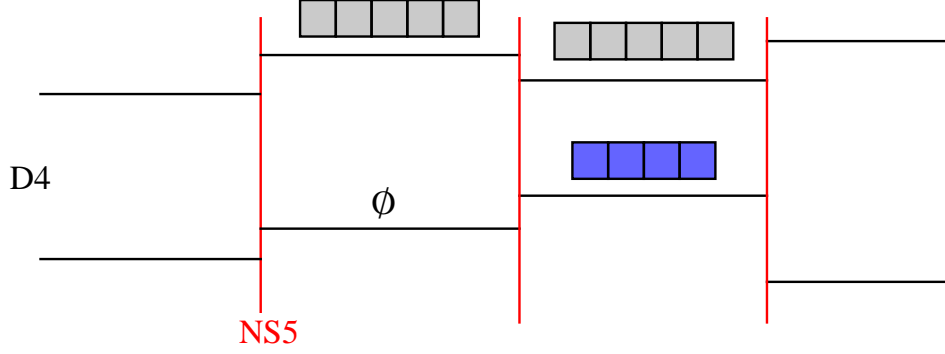


Figure 3.2: $SU(2)$ vortex from quiver theory

Let's focus on $Z_{\text{bifund}}^{(1,1)}$:

$$Z_{\text{bifund}}^{(1,1)} = \prod_{s \in Y_1} (\epsilon_2 (A_{Y_1}(s) + L_{W_1}(s) + 1)) \prod_{t \in W_1} (-\epsilon_2 (A_{W_1}(t) + L_{Y_1}(t) + 1))$$

where:

$$\prod_{t \in W_1} (-\epsilon_2 (A_{W_1}(t) + L_{Y_1}(t) + 1)) = \prod_{(i,j) \in W_1} -\epsilon_2 (A_{W_1}(i, j) + L_{Y_1}(i, j) + 1) \quad (3.1.8)$$

From the discussion of the previous section we know that the choice of the fundamental mass parameter in (3.1.7) implies that Y_1 is a row Young Tableau. Moreover, from the results in the Appendix A.2, one gets that the bifundamental masses in (3.1.7) set also W_1 to be a row of the same length which we call k_1 , see the Figure 3.2. To simplify the formulae, let's define some notations

$$(x)_{Y,W} := \prod_{s \in Y} \prod_{t \in W} (x + \epsilon_2 (A_Y(s) + L_W(s) + 1)) (x - \epsilon_2 (A_W(t) + L_Y(t) + 1)) ,$$

and

$$\begin{aligned} (x)_Y &:= (x)_{Y, \emptyset} , \\ H_Y &:= (0)_{Y, \emptyset} , \\ (x)_k &:= (x)_{\emptyset, (1^k)} , \\ (x)_{k_1, k_2} &:= (x)_{(1^{k_1}), (1^{k_2})} . \end{aligned}$$

Let's calculate $Z_{\text{bifund}}^{(1,1)}$ explicitly and the result is

$$Z_{\text{bifund}}^{(1,1)} = \prod_{i=1}^{k_1} \epsilon_2 i \prod_{j=1}^{k_1} -\epsilon_2 j = (\epsilon_2)_{k_1}^2 (-1)^{k_1} . \quad (3.1.9)$$

The contribution form $Z_{\text{bifund}}^{(2,2)}$ is instead

$$Z_{\text{bifund}}^{(2,2)} = \prod_{t \in W_2} -\epsilon_2 (A_{W_2}(t) + L_\emptyset(t)) = \prod_{(i,j) \in W_2} -\epsilon_2 (j - i - 1) , \quad (3.1.10)$$

which is non zero only if W_2 is a row. Let's denote its length by k_2 . Then

$$Z_{\text{bifund}}^{(2,2)} = (\epsilon_2)_{k_2} . \quad (3.1.11)$$

By including the contributions from $Z_{\text{bifund}}^{(1,2)}$ and $Z_{\text{bifund}}^{(2,1)}$ we get the final formula

$$Z_{\text{bifund}} = (\epsilon_2)_{k_1}^2 (-1)^{k_1} (\epsilon_2)_{k_2} (-2a)_{k_1} (2a + \epsilon_2)_{k_1, k_2} . \quad (3.1.12)$$

The contribution from the antifundamental matter can be computed with analogous methods giving

$$Z_{\text{antifund}} = (\epsilon_2)_{k_1} (-2a)_{k_1} . \quad (3.1.13)$$

The generic contributions from the vector multiplets are

$$\begin{aligned} Z_{\text{vect}}(a, Y) &= \prod_{\alpha, \beta=1}^2 Z_{\text{vect}}^{(\alpha, \beta)}(a, Y) , \\ Z_{\text{vect}}^{(\alpha, \beta)}(a, Y) &= \prod_{s \in Y_\alpha} \prod_{t \in Y_\beta} (a_{\alpha, \beta} + \epsilon_2 (A_{Y_\alpha}(s) + L_{Y_\beta}(s) + 1))^{-1} \\ &\quad (a_{\alpha, \beta} - \epsilon_2 (A_{Y_\beta}(t) + L_{Y_\alpha}(t) + 1))^{-1} , \\ a_{\alpha, \beta} &:= a_\alpha - a_\beta , \end{aligned} \quad (3.1.14)$$

which are reduced at the first node of of the quiver diagram to

$$Z_{\text{vect}}(a, Y) = ((\epsilon_2)_{k_1} (-2a)_{k_1})^{-2} . \quad (3.1.15)$$

The fundamental matter Z_{fund} is of the standard form

$$Z_{\text{fund}} = (\tilde{a} + \mu_3)_{k_1} (\tilde{a} + \mu_4)_{k_2} (-\tilde{a} + \mu_3)_{k_1} (-\tilde{a} + \mu_4)_{k_2} , \quad (3.1.16)$$

while the contribution from the second gauge factor of the quiver is

$$Z_{\text{vect}}(\tilde{a}, W) = \frac{(-1)^{k_1+k_2}}{(\epsilon_2)_{k_1}^2 (\epsilon_2)_{k_2}^2 (2a + \epsilon_2)_{k_1, k_2} (-2\tilde{a})_{k_2, k_1}} . \quad (3.1.17)$$

In summary, the total partition function of the quiver theory with specific choice of masses reads

$$Z_{\text{Quiver}}(k_1, k_2) = \frac{(-1)^{k_1} (\tilde{a} + \mu_3)_{k_1} (\tilde{a} + \mu_4)_{k_2} (-\tilde{a} + \mu_3)_{k_1} (-\tilde{a} + \mu_4)_{k_2}}{(\epsilon_2)_{k_1} (\epsilon_2)_{k_2} (2\tilde{a})_{k_1, k_2}} . \quad (3.1.18)$$

This, up-to a sign factor which can be absorbed in redefining the vortex counting parameter coincides to¹ the $SU(2)$ vortex partition function studied in chapter two,

$$Z_{\text{vortex}}^{\text{SU}(2)}(\mathbf{k}) = \frac{(-1)^{k_2} (a - m_1)_{k_1} (-a - m_1)_{k_2} (a - m_2)_{k_1} (-a - m_2)_{k_2}}{(\epsilon_2)_{k_1} (\epsilon_2)_{k_2} (a_{1,2})_{k_1, k_1}} . \quad (3.1.19)$$

Notice that we should identify $m_i = -\mu_{i+2}$, and \tilde{a} as a , since it is the second gauge factor that couples to hypermultiplets with generic masses.

To conclude the matching, notice that in the two-node quiver theory, we have two parameters q_1, q_2 which are the exponential of the gauge couplings of the quiver theory. These are related to the vortex counting parameters z_1, z_2 of vortex partition functions as

$$q_1^{k_1} (q_2)^{k_1+k_2} = (q_1 q_2)^{k_1} q_2^{k_2} = z_1^{k_1} z_2^{k_2} . \quad (3.1.20)$$

From the CFT viewpoint z_i are the insertion points of the degenerate fields.

3.1.3 $SU(2)$ Simple Surface Operators

A natural question is to find what's the result in the other channel. As expected we find it is the result of [9]. So, let's now choose $s_2 = 1$, then

$$\begin{aligned} \tilde{a} &= a - \frac{\epsilon_2}{2} , \\ m_{1,1} &= \epsilon_2 , \\ m_{2,2} &= 0 , \\ m_{1,2} &= 2a , \\ m_{2,1} &= -2a + \epsilon_2 . \end{aligned} \quad (3.1.21)$$

¹With respect to [?] we set $\hbar = -\epsilon_2$. These sign factors will be disregarded in the following without further notice.

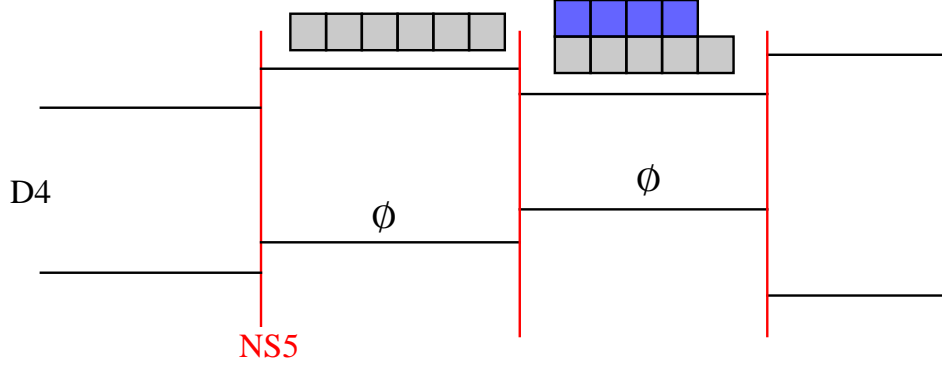


Figure 3.3: $SU(2)$ simple surface operators from quiver theory

In this case, the contribution of the bifundamentals reads

$$Z_{\text{bifund}}^{(1,1)} = \prod_{s \in Y_1} (\epsilon_2(A_{Y_1}(s) + L_{W_1}(s) + 2)) \prod_{t \in W_1} (-\epsilon_2(A_{W_1}(t) + L_{Y_1}(t))) , \quad (3.1.22)$$

where

$$\prod_{t \in W_1} (-\epsilon_2(A_{W_1}(t) + L_{Y_1}(t))) = \prod_{(i,j) \in W_1} -\epsilon_2(A_{W_1}(i,j) + L_{Y_1}(i,j)) .$$

Using once again the results in the Appendix A.2, the bifundamental contribution,

$$Z_{\text{bifund}}^{(2,2)} = \prod_{t \in W_2} (-\epsilon_2(A_{W_2}(t) + L_{\emptyset}(t) + 1)) = \prod_{t \in W_2} (-\epsilon_2(j - i)) , \quad (3.1.23)$$

is non vanishing only if $W_2 = \emptyset$, see Figure 3.3.

Therefore, the bifundamental contributions are given by

$$\begin{aligned} Z_{\text{bifund}}^{(1,1)} &= H_{Y_1} H_{W_1} (-1)^{k_1} , \\ Z_{\text{bifund}}^{(2,2)} &= 1 , \\ Z_{\text{bifund}}^{(1,2)} &= (-1)^{k_1+1} (-2a)_{k_1+1} , \\ Z_{\text{bifund}}^{(2,1)} &= (-1)^{k_1+k_2} (\tilde{a}_{1,2})_{W_1} . \end{aligned}$$

The contribution from the other factors can be analogously derived to be

$$Z_{\text{antifund}} Z_{\text{vect}}(a, Y) = \frac{1}{(\epsilon_2)_{k_1+1} (-2a)_{k_1+1}} , \quad (3.1.24)$$

$$Z_{\text{vect}}(\tilde{a}, W) = \frac{1}{\left(H_{W_1}(\tilde{a}_{1,2})_{W_1}\right)^2}, \quad (3.1.25)$$

$$Z_{\text{fund}} = (\tilde{a} - \mu_3)_{W_1} (\tilde{a} - \mu_4)_{W_1}, \quad (3.1.26)$$

and finally we get

$$Z_{\text{Quiver}}(W_1) = \frac{(-1)^{k_1+k_2+1} (\tilde{a} - \mu_3)_{W_1} (\tilde{a} - \mu_4)_{W_1}}{H_{W_1}(\tilde{a}_{1,2})_{W_1}}, \quad (3.1.27)$$

which is the partition function of $SU(2)$ simple surface operator [9]

$$Z_{\text{simple surface}} = \frac{(a + m_1)_{W_1} (a + m_2)_{W_1}}{H_{W_1}(a_{1,2})_{W_1}}. \quad (3.1.28)$$

Now the identification of counting parameters is

$$q_1^{k_1+1} (q_2)^{k_1+k_2} = \frac{z_1}{z_2} z_1^{k_1} z_2^{k_2}. \quad (3.1.29)$$

As already noticed, z_i are the insertion points of the degenerate fields.

3.1.4 Relation to Open Topological String Amplitudes

The amplitudes discussed in the previous sections can be derived as four dimensional limits of open topological string amplitudes on the strip with suitable boundary conditions [9, 11].

The discussion of the previous section provides the CFT interpretation of this class of strip amplitudes, as summarized in Figure 3.4 and Figure 3.5. Actually, this is the simplest situation. For example we can have more than two degenerate states, then does this story still holds? The answer is yes. From our previous calculations, we can deduce three general laws:

1. the number of nodes of the quiver equals the number of degenerate states.
2. the total number of rows of Young-tableaux increase by one when counting from left to right along the quiver of gauge theory nodes.
3. different fusion channels just tell us on which gauge factor of the quiver to associate an extra row in the partition.

So if we have n degenerate states, the corresponding quiver has n nodes, and on each node there are two choices to add a new row. For convenience let's define a fusion vector $\mathfrak{V} \in \mathbb{Z}_2^n$, whose i -th component is 1 if we add a new row onto the partition attached to the first $D4$ brane

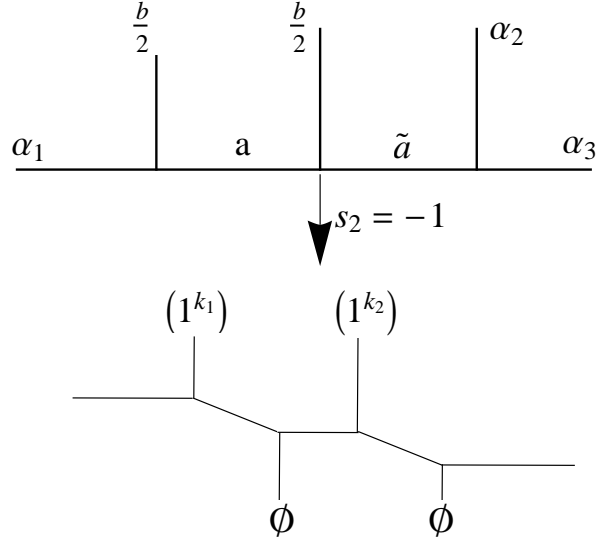


Figure 3.4: relation between $SU(2)$ vortex and CFT

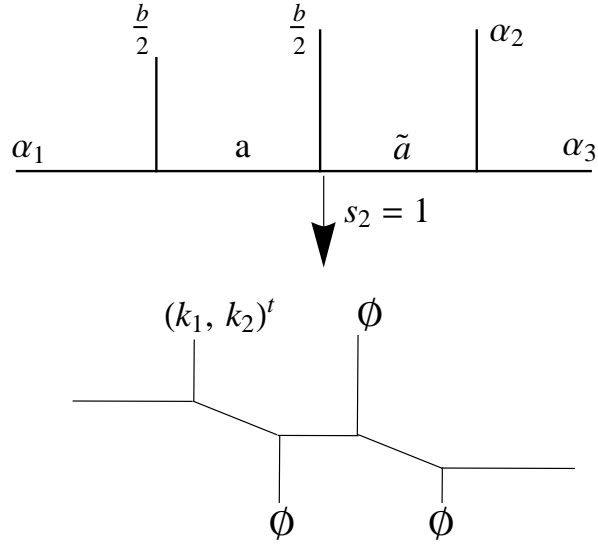


Figure 3.5: relation between $SU(2)$ simple surface operator and CFT

and 2 if to the second. For example, the nonabelian vortex partition function is associated to $\mathfrak{V} = (1, 2)$, while the simple surface operator partition function is associated to $\mathfrak{V} = (1, 1)$.

When we have n degenerate states, the Young-tableaux on the final node are a couple (Y, W) satisfying the constraint $n_1 + n_2 = n$, where n_1, n_2 are respectively the number of rows of Y and of W . Hence we conclude that the four dimensional limit of the strip amplitudes of

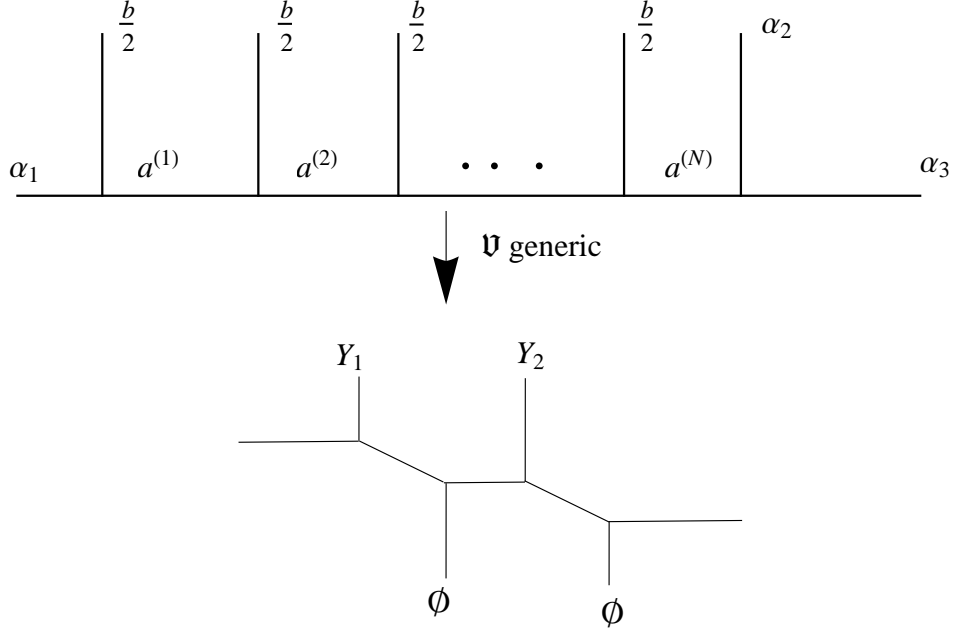


Figure 3.6: relation between $SU(2)$ strip amplitudes and CFT

the form $A_{\{\emptyset, \emptyset\}}^{\{Y, W\}}$, that is with boundary conditions labeled by Y and W , reproduces the full conformal block vector space including all the possible fusion channels. For example we can choose $\mathfrak{V} = (1, \dots, 1, 2, \dots, 2)$, where there are n_1 1's and n_2 2's and can prove explicitly that for this choice of fusion vector our claim is correct, see Figure 3.6.

3.2 The $SU(N)$ Generalization

In the following we will give the natural generalization to $SU(N)$ theories. We know that the $SU(N)$ vortex partition function should have N independent counting parameters, thus from the previous section's discussion we know that the associated $SU(N)$ quiver theory will have N nodes. The quiver configuration reads as the brane construction illustrated in Figure 3.7.

3.2.1 $SU(N)$ Vortices

The Young-tableaux configuration of quiver gauge theory corresponding to vortex partition function is such that at the L -th node the arrows of Young tableaux are

$$Y^{(L)} = (1^{k_1}, \dots, 1^{k_L}, \emptyset, \dots, \emptyset) .$$

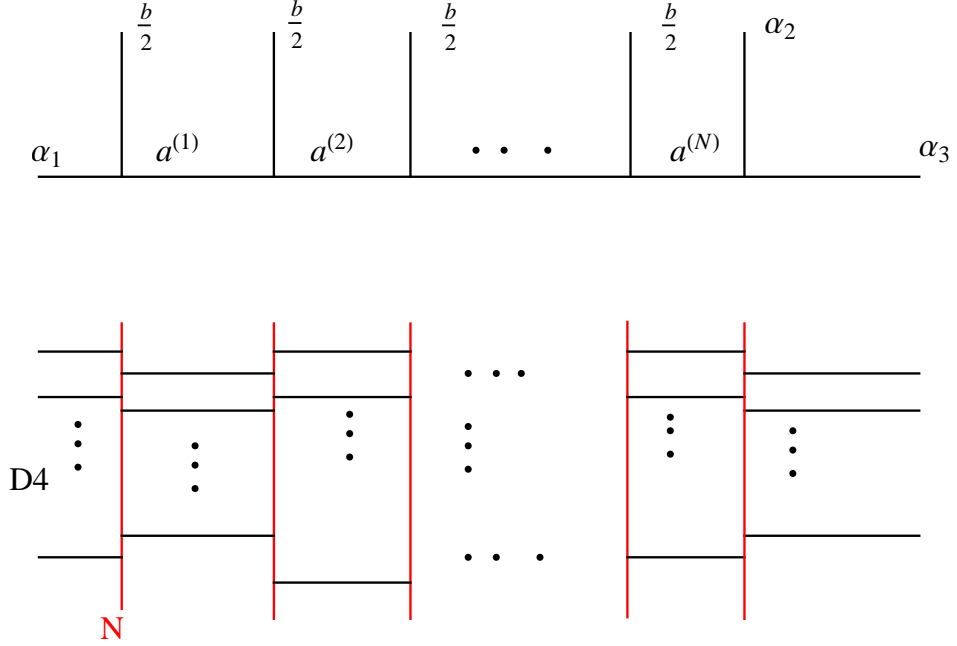


Figure 3.7: AGT relation between $SU(N)$ quiver gauge theory and CFT

This configuration can be obtained from a given bifundamental mass assignments as displayed in the (3.2.1). We will see that this choice of masses correctly reproduces the fusion rules for Toda field theory.

Let us consider the L -th node of the quiver and calculate $Z_L Z_{L,L+1}$, where Z_L is the vector contribution of the L -th node while $Z_{L,L+1}$ the corresponding bifundamental. Following the arguments in the Appendix A.2, we can read out the L -th bifundamental mass to be

$$m_{\alpha,\alpha}^{(L)} := a_{\alpha}^{(L)} - a_{\alpha}^{(L+1)} - m_L = \delta_{\alpha,L+1} \epsilon_2. \quad (3.2.1)$$

Then the matrix of masses is given by

$$m_{\alpha,\beta}^{(L)} = \begin{cases} a_{\alpha,\beta}^{(L)} = a_{\alpha,\beta}^{(L+1)} & \alpha \in [1, L]; \beta = [1, L] \\ a_{\alpha,\beta}^{(L+1)} & \alpha \in [1, L]; \beta \in [L+1, N] \\ a_{\alpha,\beta}^{(L)} & \alpha \in [L+1, N]; \beta = [1, L] \end{cases}.$$

We find it better to write Z_L in three parts according to above mass matrix formula

$$Z_L^{-1} = \prod_{\alpha,\beta=1}^L \left(a_{\alpha,\beta}^{(L)} \right)_{k_{\alpha}, k_{\beta}} \prod_{\alpha=1}^L \prod_{\beta=L+1}^N (-1)^{k_{\alpha}} \left(a_{\beta,\alpha}^{(L)} \right)_{k_{\alpha}} \prod_{\beta=1}^L \prod_{\alpha=L+1}^N \left(a_{\alpha,\beta}^{(L)} \right)_{k_{\beta}}.$$

Correspondingly, $Z_{L,L+1}$ read

$$Z_{L,L+1} = \left\{ \prod_{\alpha=1}^L \prod_{\beta=1}^L \left(m_{\alpha,\beta}^{(L)} \right)_{k_{\alpha},k_{\beta}} \right\} \times \\ \left\{ \prod_{\alpha=1}^L \prod_{\beta=L+2}^N (-1)^{k_{\alpha}} \left(-m_{\alpha,\beta}^{(L)} \right)_{k_{\alpha}} \right\} \left\{ \prod_{\alpha=1}^L \left(m_{\alpha,L+1}^{(L)} \right)_{k_{\alpha},k_{L+1}} \right\} \times \\ \left\{ \prod_{\alpha=L+1}^N \prod_{\beta=1}^L \left(m_{\alpha,\beta}^{(L)} \right)_{k_{\beta}} \right\} \left\{ \prod_{\alpha=L+1}^N \left(m_{\alpha,L+1}^{(L)} \right)_{k_{L+1}} \right\} .$$

Then we get:

$$Z_L Z_{L,L+1} = \frac{\left\{ \prod_{\alpha=1}^L \left(a_{\alpha,L+1}^{(L+1)} \right)_{k_{\alpha},k_{L+1}} \right\}}{\left\{ \prod_{\alpha=1}^L (-1)^{k_{\alpha}} \left(a_{L+1,\alpha}^{(L)} \right)_{k_{\alpha}} \right\}} \left\{ \prod_{\alpha=L+2}^N \left(a_{\alpha,L+1}^{(L+1)} \right)_{k_{L+1}} \right\} (\epsilon_2)_{k_{L+1}} . \quad (3.2.2)$$

The mass spectrum of the antifundamental hypermultiplets is assigned as

$$(\mu_1, \mu_2, \dots, \mu_N) = \left(-a_1^{(1)} - \epsilon_2, -a_2^{(1)}, \dots, -a_N^{(1)} \right) . \quad (3.2.3)$$

and the correspondent contribution to the instanton partition function is

$$Z_{\text{antifund}} = \prod_{f=1}^N \prod_{i=1}^{k_1} \left(a_1^{(1)} + \mu_f + \epsilon_2(1-i) \right) = (-1)^{Nk_1} (\epsilon_2)_{k_1} \prod_{i=2}^N \left(a_{i,1}^{(1)} \right)_{k_1} . \quad (3.2.4)$$

Finally, the vector contribution of the last N -th node is

$$Z_N^{-1} = \prod_{\alpha=1}^N (\epsilon_2)_{k_{\alpha}}^2 (-1)^{k_{\alpha}} \prod_{\alpha < \beta}^N (-1)^{k_{\alpha}+k_{\beta}} \left(a_{\alpha,\beta}^{(N)} \right)_{k_{\alpha},k_{\beta}}^2 . \quad (3.2.5)$$

Then the instanton partition function of this quiver is:

$$Z_{\text{Quiver}} = \frac{(-1)^{Nk_1 + \sum_{\alpha} k_{\alpha}} \prod_{\alpha,f=1}^N (-1)^{k_{\alpha}} \left(-a_{\alpha}^{(N)} + \mu_{f+N} \right)_{k_{\alpha}}}{\prod_{\alpha=1}^N (\epsilon_2)_{k_{\alpha}} \prod_{\alpha < \beta}^N (-1)^{k_{\beta}} \left(a_{\alpha,\beta}^{(N)} \right)_{k_{\alpha},k_{\beta}}} . \quad (3.2.6)$$

Following the result of chapter two, and identifying $\hbar = -\epsilon_2$, the $SU(N)$ vortex partition

function can be written as

$$\begin{aligned}
Z_{\text{vortex}}^{\text{SU}(N)} &= \sum_{\mathbf{k}} Z_{\text{vortex}}^{\text{SU}(N)}(\mathbf{k}) \prod_{i=1}^N z_i^{k_i}, \\
Z_{\text{vortex}}^{\text{SU}(N)}(\mathbf{k}) &= \frac{\prod_{\alpha, f=1}^N (-1)^{k_\alpha} (-a_\alpha - m_f)_{k_\alpha}}{\prod_{\alpha=1}^N (\epsilon_2)_{k_\alpha} \prod_{\alpha < \beta}^N (-1)^{k_\beta} (a_{\alpha, \beta})_{k_\alpha, k_\beta}}.
\end{aligned} \tag{3.2.7}$$

This can be identified with the quiver instanton partition function by setting $a_\alpha^{(N)} = a_\alpha$ and $m_f = -\mu_{N+f}$. The counting parameters z_i are identified as

$$\begin{aligned}
\prod_{i=1}^N q_i^{\sum_{j=1}^i k_j} &= \prod_{i=1}^N z_i^{k_i}, \\
z_i &:= \prod_{j=i}^{N+1-i} q_j.
\end{aligned} \tag{3.2.8}$$

In conclusion, the instanton partition function of quiver gauge theory with

$$Y^{(L)} = (1^{k_1}, \dots, 1^{k_L}, \emptyset, \dots, \emptyset)$$

with parameters in formula (3.2.1) and (3.2.3) gives the $SU(N)$ vortex partition function.

3.2.2 $SU(N)$ Simple Surface Operators

From the previous arguments we can argue that the four dimensional limit of the strip amplitude $A_{\{\emptyset, \dots, \emptyset\}}^{W, \emptyset, \dots, \emptyset}$, with $W = (k_1, k_2, \dots, k_N)$, corresponds to the quiver gauge theory with the following Young-tableaux assignments

$$\begin{aligned}
Y^{(L)} &= (Y_L, \emptyset, \dots, \emptyset), \\
Y_L^t &= (k_1 + (N - L), k_2 + (N - L), \dots, k_L + (N - L)).
\end{aligned} \tag{3.2.9}$$

The corresponding bifundamental masses can be obtained by following the arguments displayed in Appendix A.2 to be

$$m_{\alpha, \alpha}^{(L)} = a_\alpha^{(L)} - a_\alpha^{(L+1)} - m_L = \delta_{\alpha, 1} \epsilon_2, \tag{3.2.10}$$

for the L -th node. The corresponding vector contribution for the L -th node is

$$Z_L^{-1} = (-1)^{|Y_L|} H_{Y_L}^2 \prod_{\beta=2}^N \left(a_{1,\beta}^{(L)} \right)_{Y_L} \prod_{\alpha=2}^N (-1)^{|Y_L|} \left(a_{1,\alpha}^{(L)} \right)_{Y_L}, \quad (3.2.11)$$

while the bifundamental is

$$Z_{L,L+1} = (\epsilon_2)_{Y_L, Y_{L+1}} \prod_{\beta=2}^N \left(a_{1,\beta}^{(L)} \right)_{Y_L} \prod_{\alpha=2}^N (-1)^{|Y_{L+1}|} \left(a_{1,\alpha}^{(L+1)} \right)_{Y_{L+1}}, \quad (3.2.12)$$

so that

$$\prod_{L=1}^{N-1} Z_L Z_{L,L+1} = \left\{ \prod_{L=1}^{N-1} \frac{(\epsilon_2)_{Y_L, Y_{L+1}}}{(-1)^{|Y_L|} H_{Y_L}^2} \right\} \frac{\prod_{\alpha=2}^N (-1)^{|Y_N|} \left(a_{1,\alpha}^{(N)} \right)_{Y_N}}{\prod_{\alpha=2}^N (-1)^{|Y_1|} \left(a_{1,\alpha}^{(1)} \right)_{Y_1}}. \quad (3.2.13)$$

Using the result of the factorization formula in Appendix A.2, we can rewrite

$$(\epsilon_2)_{Y_L, Y_{L+1}} = (-1)^{|Y_L|+L} H_{Y_L} H_{Y_{L+1}}, \quad (3.2.14)$$

and finally get

$$\prod_{L=1}^{N-1} Z_L Z_{L,L+1} = (-1)^{\sum_{L=1}^{N-1} L} \frac{H_{Y_N} \prod_{\alpha=2}^N (-1)^{|Y_N|} \left(a_{1,\alpha}^{(N)} \right)_{Y_N}}{H_{Y_1} \prod_{\alpha=2}^N (-1)^{|Y_1|} \left(a_{1,\alpha}^{(1)} \right)_{Y_1}}. \quad (3.2.15)$$

Notice that, as in the $SU(2)$ case, the spectrum of antifundamental hypermultiplets is fixed to be the same for both simple surface operator and nonabelian vortices. What distinguishes the different cases are the different fusion rules channels. The corresponding factors are then

$$Z_{\text{fund}} = \prod_{f=1}^N \left(a_1^{(N)} - \mu_{f+N} \right)_{Y_N}, \quad (3.2.16)$$

$$Z_{\text{antifund}} = (-1)^{|Y_1|} H_{Y_1} \prod_{\alpha=2}^N \left(a_{1,\alpha}^{(1)} \right)_{Y_1}, \quad (3.2.17)$$

$$Z_N^{-1} = (-1)^{N|Y_N|} H_{Y_N}^2 \prod_{\alpha=2}^N (a_{1,\alpha})_{Y_N}^2, \quad (3.2.18)$$

which finally give

$$Z_{\text{Quiver}} = (-1)^{\sum_{L=1}^N L+N|Y_1|+|Y_N|} \frac{\prod_{f=1}^N \left(a_1^{(N)} - \mu_{f+N} \right)^{Y_N}}{H_{Y_N} \prod_{\alpha=2}^N \left(a_{1,\alpha}^{(N)} \right)^{Y_N}}. \quad (3.2.19)$$

This, after the identifications $\hbar = -\epsilon_2$, $a_1^{(N)} = a_1$, $m_f = -\mu_{f+N}$ and $\lambda = Y_N$, is the simple surface operator partition function discussed in [9] under the same counting parameters identification that we used in the last section.

3.2.3 Toda Fusion Rules from Quiver Gauge Theory

In this subsection we show how to *derive* fusion rules of semidegenerate states of Toda field theory from our construction. Let's concentrate on the L -th node of the quiver and recall the diagonal part of the mass assignment

$$m_{\alpha,\alpha}^{(L)} := a_{\alpha}^{(L)} - a_{\alpha}^{(L+1)} - m_L. \quad (3.2.20)$$

By denoting $\mathbf{m}_L = m_L(1, 1, \dots, 1)$, being a vector of N entries all equal to m_L , we can write the above formula as

$$\mathbf{a}^{(L)} - \mathbf{a}^{(L+1)} = \mathbf{m}^{(L)} - \mathbf{m}_L, \quad (3.2.21)$$

where $\mathbf{a}^{(L)}$ denotes the vector of internal momenta at the L -th node and $\mathbf{m}^{(L)}$ the vector of diagonal entries of the mass matrix at the L -node. Actually, for this assignment of external momenta, Toda fusion rules have N channels. For the i -th channel $\mathbf{m}^{(L)} = \epsilon_2 u_i = \epsilon_2 \left(u_1 - \sum_{j=1}^{i-1} e_j \right)$. Where u_i is the unit vector in the i -th direction in \mathbb{R}^N and $e_j := u_j - u_{j+1}$ are the simple roots of the \mathfrak{sl}_N algebra. Then we have

$$\mathbf{a}^{(L)} - \mathbf{a}^{(L+1)} = \epsilon_2 \left(u_1 - \sum_{j=1}^{i-1} e_j \right) - \mathbf{m}_L = \epsilon_2 \left(u_1 - \frac{\mathbf{m}_L}{\epsilon_2} \right) - \epsilon_2 \sum_{j=1}^{i-1} e_j. \quad (3.2.22)$$

If we set $\mathbf{m}_L = \epsilon_2 \frac{1}{N}(1, 1, \dots, 1)$, then

$$\mathbf{a}^{(L)} - \mathbf{a}^{(L+1)} = \epsilon_2 (-\omega_1) - \epsilon_2 \sum_{j=1}^{i-1} e_j, \quad (3.2.23)$$

where ω_1 is the highest weight of the fundamental representation of \mathfrak{sl}_N . The above formula can be recognized as the fusion rule calculated in [62].

For $SU(N)$ N -node quiver, we can have N semidegenerate states, for each one of them

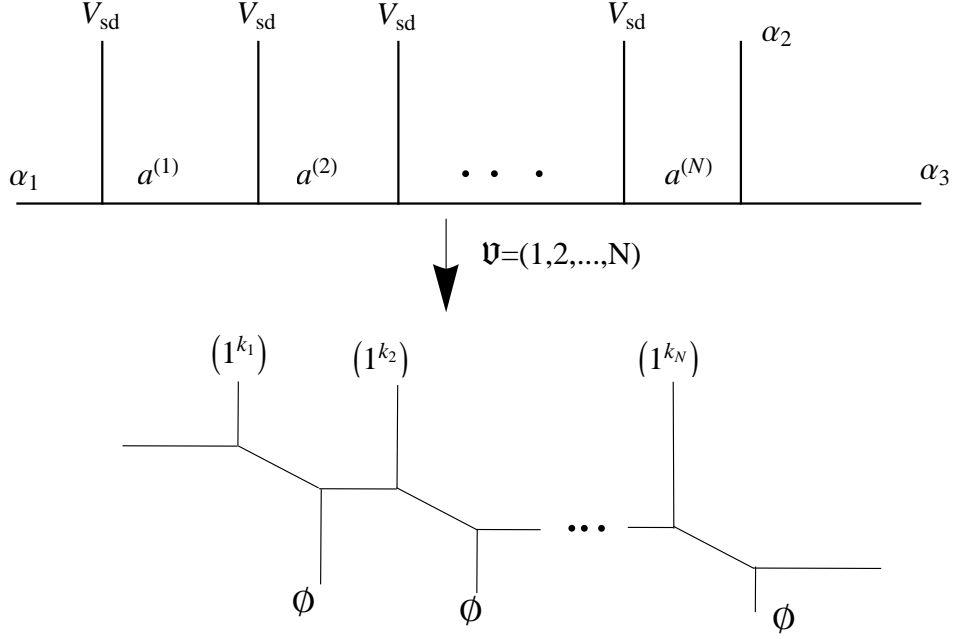


Figure 3.8: relation between $SU(N)$ vortex and CFT

we have N channels. We can use an N -dimensional vector of integer entries \mathfrak{V} to denote the choice of the fusion channels. The fusion vector \mathfrak{V} is built as follows: if on the L -th node we choose k -th channel, namely $m_{\alpha,\alpha}^{(L)} = \epsilon_2 \delta_{\alpha,k}$, then the corresponding L -th component of \mathfrak{V} is set equal to k . For example for the $SU(N)$ vortex $\mathfrak{V}_{\text{vortex}} = (1, 2, \dots, N)$, while for $SU(N)$ simple surface operator, $\mathfrak{V}_{\text{simple surface}} = (1, 1, \dots, 1)$.

The relation with the four dimensional limit of strip amplitudes goes as in the $SU(2)$ as depicted in Figure 3.8, 3.9, and 3.10.

Notice that the four dimensional limit of strip amplitudes correspond to conformal blocks with only two independent external momenta, and one independent internal momentum. The number of degenerate states inserted in the conformal block corresponds to the total number of rows of the Young tableaux parametrizing the open string boundary conditions. This suggests that in order to have arbitrary boundary conditions one should consider conformal blocks with an arbitrary number of degenerate field insertions. Since we know that the full instanton partition function can be obtained by gluing two strip amplitudes with generic boundary conditions, this would provide a conformal field theory picture of this operation. From the CFT viewpoint, the infinite number of degenerate insertions could condense in a line operator [83] which could be used to glue the two CFT amplitudes to obtain the full result.

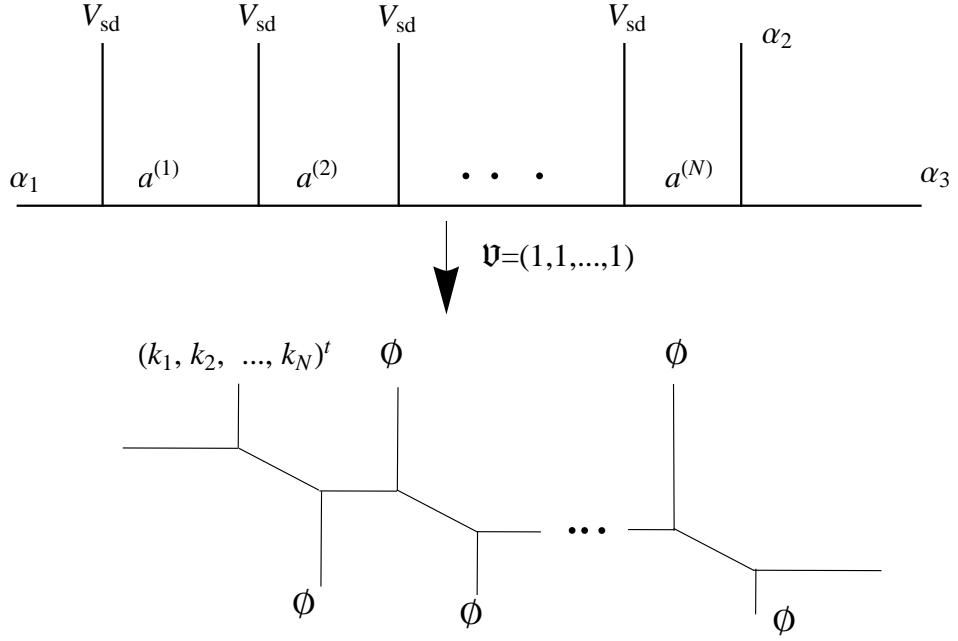


Figure 3.9: relation between $SU(N)$ simple surface operators and CFT

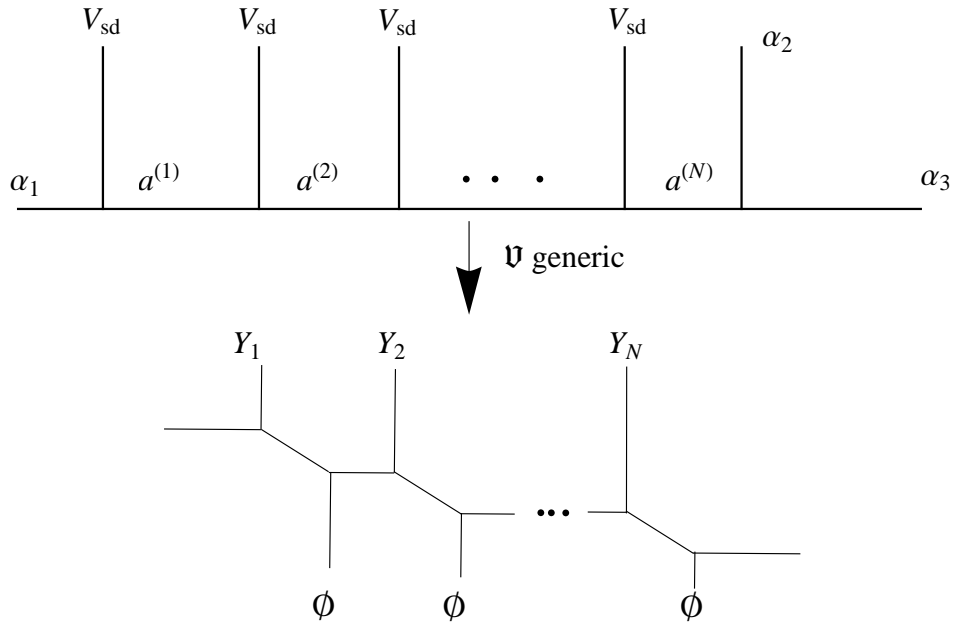


Figure 3.10: relation between $SU(N)$ strip amplitudes and CFT

Chapter 4

Orbifold Vortex and Super Liouville Theory

An important generalization of the AGT duality is the correspondence between ALE instanton counting and the conformal blocks of $\mathcal{N} = 1$ super Liouville theory, as has been studied in [42, 43, 44, 45], [46, 47, 48, 49] and [50]. The $\mathcal{N} = 1$ super Liouville theory has two dynamic fields, ϕ and its superpartner ψ . The action is

$$S = \int d^2z \left\{ \frac{1}{2\pi} ((\partial\phi)^2 + \psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}) + 2i\mu b^2 \bar{\psi}\psi e^{b\phi} + 2\pi b^2 \mu^2 e^{2b\phi} \right\}.$$

Due to the super symmetry there are two types of primary fields, $V_\alpha^R = \sigma^\pm e^{\alpha\phi}$ and $V_\alpha^{NS} = e^{\alpha\phi}$. R and NS in the superscript denote the NS and R sector of super Virasoro algebra respectively. σ^\pm are the twist operators with the conformal weight $\frac{1}{16}$ and they are defined to satisfy

$$\psi(z)\sigma^\pm(0) \simeq \frac{\sigma^\mp(0)}{\sqrt{2z}}.$$

We also assume that σ^+ commute and σ^- anticommute with fermions. An important fact is that the correlation functions in the NS sector are closed by itself, while the R sector correlation functions are not closed by itself. This is because among the three-point functions of primary states, only $\langle V_{\alpha_1}^{NS} V_{\alpha_2}^{NS} V_{\alpha_3}^{NS} \rangle$ and $\langle V_{\alpha_1}^R V_{\alpha_2}^R V_{\alpha_3}^{NS} \rangle$ do not vanish. More knowledge on super Liouville theories can be find in [84] and references there in.

In this chapter we generalize results of previous chapters to the case of instantons/vortices on orbifolds and their conformal theory duals. Again, using localization techniques, we can obtain orbifold vortex partition functions, and then we study the degeneration phenomenon of orbifold quiver instanton partition functions. This will not only tell us how to extract

orbifold vortex partition functions from that of instantons but also give information about surface operators of orbifold gauge theories.

After studying the relation between orbifold vortices and orbifold instantons, one is urged to study the AGT dual of orbifold vortices. It is difficult to find the CFT dual of vortex partition functions directly. The trick here is that we can use four dimensional gauge theories as a bridge connecting conformal field theories and vortex theories as we did in previous chapters. The AGT dual of correlation functions of pure NS primary fields was studied in [42, 43, 44, 45] and that of Whittaker vectors in the Ramond sector was studied in [50]. However, our analysis shows that in order to find the AGT dual of orbifold vortices, it is necessary to have a complete knowledge of the AGT duality of super Liouville theory with both NS and Ramond sectors. We study the super Liouville theory dual of orbifold vortices based on known results about correlation functions of degenerate fields in Ramond sector [85], [86] and show that orbifold vortex partition functions can be identified with correlation functions of lowest degenerate states in the Ramond sector.

At first, we quickly review instanton counting on $\mathbb{C}^2/\mathbb{Z}_p$. Then we calculate vortex partition function on \mathbb{C}/\mathbb{Z}_p . After studying the degeneration phenomenon of orbifold instanton partition functions, we will show the instanton/vortex relation on orbifolds. Finally we will study the CFT dual of vortex partition function on \mathbb{C}/\mathbb{Z}_p .

4.1 Instantons on $\mathbb{C}^2/\mathbb{Z}_p$

In this section we will review how to do instanton counting for $U(N)$ linear quiver gauge theory on the orbifold space \mathbb{C}^2/Γ , where $\Gamma = \mathbb{Z}_p$. [87] is a standard reference for this topic. We use k to denote the instanton number and parameters for pure instanton counting are Coulomb branch parameters a_α where α runs from 1 to N and the Ω -deformation parameters, ϵ_1, ϵ_2 . Due to the orbifold action, $a_\alpha, \epsilon_1, \epsilon_2$ have respectively discrete charges $q_\alpha, 1, -1$. Notice that discrete charges take value in \mathbb{Z}_p , so two charges are the same if they are congruent modulo p . Since \mathbb{Z}_p commutes with the gauge groups, under this assignment of charges, the gauge groups will break in the following way,

$$U(N) \longrightarrow \prod_q U(n_q) ,$$

$$n_q = \sum_{\alpha} \delta_{q, q_\alpha} .$$

It seems that Γ will change the fixed point structure of instanton counting drastically, but due to the fact that $\Gamma \in U(1)^2$ of the localization torus action, fixed points are still characterized

by N Young tableaux of total number of boxes equals to k . Similarly the auxiliary $U(k)$ group will also break as

$$U(k) \longrightarrow \prod_q U(k_q) .$$

As we know each box in a Young tableau represents an instanton, and the corresponding discrete charge is just $q_\alpha + i - j$ for a box at position (i, j) of the α -th Young tableau. So $k_q = \dim V_q = \text{number of instantons with discrete charge } q$. Here V and W are complex linear spaces of dimension k and N . Then we have following linear decomposition of the Euler character of the tangent bundle of instanton moduli space

$$\begin{aligned} \chi_\Gamma &= V^* \otimes V (T_1 + T_2 - 1 - T_1 T_2) + W^* \otimes V + V^* \otimes W T_1 T_2 \\ &= \sum_q (V_q^* V_{q+1} + V_{q+1}^* V_q - V_q^* V_q T_1 T_2 - V_q^* V_q + W_q^* V_q + V_q^* W_q T_1 T_2) , \\ V_q &= \sum_{\alpha=1}^N \sum_{s \in Y_\alpha} T_{a_\alpha} T_1^{-j_s+1} T_2^{-i_s+1} \delta_{q_\alpha+i_s-j_s, q} , \\ W_q &= \sum_{\alpha=1}^N T_{a_\alpha} \delta_{q_\alpha, q} . \end{aligned} \tag{4.1.1}$$

After some algebra we get

$$\begin{aligned} \chi_\Gamma^{\text{vector}} &= - \sum_{\alpha, \beta} \sum_{s \in Y_\alpha} \left(T_{a_{\alpha, \beta}} T_1^{-L_\beta(s)} T_2^{A_\alpha(s)+1} + T_{a_{\beta, \alpha}} T_1^{L_\beta(s)+1} T_2^{-A_\alpha(s)} \right) \delta_{L_\beta(s)+A_\alpha(s)+1, q_{\alpha, \beta}} \\ &= - \sum_{\alpha, \beta} \sum_{s \in Y_\alpha} T_{a_{\alpha, \beta}} T_1^{-L_\beta(s)} T_2^{A_\alpha(s)+1} \delta_{L_\beta(s)+A_\alpha(s)+1, q_{\alpha, \beta}} \\ &\quad - \sum_{\alpha, \beta} \sum_{t \in Y_\beta} T_{a_{\alpha, \beta}} T_1^{L_\alpha(s)+1} T_2^{-A_\beta(s)} \delta_{L_\alpha(s)+A_\beta(s)+1, q_{\beta, \alpha}} . \end{aligned} \tag{4.1.2}$$

To obtain 4d instanton partition functions, we need to set $T_1 = e^{\epsilon_1}, T_2 = e^{\epsilon_2}, T_{a_\alpha} = e^{a_\alpha}$ and then take the four dimensional limit. As we already know, vortex partition functions lie in $\epsilon_+ = \epsilon_1 + \epsilon_2 = 0$ limit of degenerate instanton partition functions, we will take this limit

in the following

$$\begin{aligned}
(Z_\Gamma^{\text{vector}}(a, Y, q_\alpha))^{-1} &= \prod_{\alpha, \beta}^N \prod_{s \in Y_\alpha} (a_{\alpha, \beta} + \epsilon_2 (A_{Y_\alpha}(s) + 1 + L_{Y_\beta}(s))) \delta_{A_{Y_\alpha}(s)+1+L_{Y_\beta}(s), q_{\alpha, \beta}} \\
&\quad \prod_{t \in W_\beta} (a_{\alpha, \beta} - \epsilon_2 (A_{Y_\beta}(t) + 1 + L_{Y_\alpha}(s))) \delta_{A_{Y_\beta}(t)+1+L_{Y_\alpha}(t), q_{\beta, \alpha}}.
\end{aligned} \tag{4.1.3}$$

Vector field contributions are in denominators of instanton partition functions, and numerators of instanton partition function will come from hypermultiplets. Because our interest lies in linear quiver gauge theories, we will only consider hypermultiplets in (anti)fundamental and bifundamental representations. Since later we will study N -node quiver gauge theory, we will take following notations

$$\begin{aligned}
\{q_\alpha^{(L)}\}_{\alpha=1}^N &: \text{the discrete charges of Coulomb branch parameters of the } L\text{-th gauge factor.} \\
q_m^{(L)} &: \text{the discrete charge of the } L\text{-th bifundamental hypermultiplet.} \\
q_\alpha^f &: \text{the discrete charge of the } \alpha\text{-th fundamental hypermultiplet.} \\
q_\alpha^{af} &: \text{the discrete charge of the } \alpha\text{-th antifundamental hypermultiplet.} \\
Q_{\alpha, \beta}^{(L)} &= q_\alpha^{(L)} - q_\beta^{(L+1)} + q_m^{(L)}.
\end{aligned} \tag{4.1.4}$$

Other notations will be used are given in Appendix A.1.

4.1.1 Adding Bifundamental Matter Fields

From the vector field contribution, we can easily obtain the contribution from bifundamental hypermultiples

$$\begin{aligned}
\chi_\Gamma^{\text{bifund}, L} &= \sum_{\alpha, \beta}^N T_{m_L} \left(\sum_{s \in Y_\alpha^{(L)}} T_{a_{\alpha, \beta}^{(L, L+1)}} T_1^{-L_{Y_\beta^{(L+1)}}(s)} T_2^{A_{Y_\alpha^{(L)}}(s)+1} \delta_{L_{Y_\beta^{(L+1)}}(s)+A_{Y_\alpha^{(L)}}(s)+1, Q_{\alpha, \beta}^{(L, L+1)}} + \right. \\
&\quad \left. \sum_{t \in Y_\beta^{(L+1)}} T_{a_{\alpha, \beta}^{(L, L+1)}} T_1^{L_{Y_\alpha^{(L)}}(t)+1} T_2^{-A_{Y_\beta^{(L+1)}}(t)} \delta_{L_{Y_\alpha^{(L)}}(t)+A_{Y_\beta^{(L+1)}}(t)+1, Q_{\beta, \alpha}^{(L, L+1)}} \right).
\end{aligned}$$

In $\epsilon_+ = 0$ limit, the contribution to instanton partition function from the L -th bifundamen-

tal hypermultiplet is

$$Z_{\Gamma}^{\text{bifund},L}(a, m, Y) = \prod_{\alpha,\beta}^N \prod_{s \in Y_{\alpha}^{(L)}} \left(m_{\alpha,\beta}^{(L)} + \epsilon_2 \left(L_{Y_{\beta}^{(L+1)}}(s) + A_{Y_{\alpha}^{(L)}}(s) + 1 \right) \right) \delta_{L_{Y_{\beta}^{(L+1)}}(s) + A_{Y_{\alpha}^{(L)}}(s) + 1, Q_{\alpha,\beta}^{(L)}} \cdot \prod_{t \in Y_{\beta}^{(L+1)}} \left(m_{\alpha,\beta}^{(L)} - \epsilon_2 \left(L_{Y_{\alpha}^{(L)}}(t) + A_{Y_{\beta}^{(L+1)}}(t) + 1 \right) \right) \delta_{L_{Y_{\alpha}^{(L)}}(t) + A_{Y_{\beta}^{(L+1)}}(t) + 1, Q_{\beta,\alpha}^{(L)}} \cdot$$

4.1.2 Adding Fundamental Matter Fields

It is easy to obtain contributions from fundamental hypermultiplets by either direct calculation or reduction from that of bifundamental hypermultiplets. The results are

$$Z_{\Gamma}^{\text{fund},q_{\beta}^f}(a, m, Y) = \prod_{\alpha=1}^N \prod_{\beta=1}^F \prod_{s \in Y_{\alpha}} (a_{\alpha} - m_{\beta} + \epsilon_1 (i_s - 1) + \epsilon_2 (j_s - 1) + \epsilon_+) \delta_{j-i, q_{\alpha} - q_{\beta}^f}, \quad (4.1.5)$$

$$Z_{\Gamma}^{\text{antifund},q_{\beta}^{af}}(a, m, Y) = \prod_{\alpha=1}^N \prod_{\beta=1}^F \prod_{s \in Y_{\alpha}} (a_{\alpha} + m_{\beta} + \epsilon_1 (i_s - 1) + \epsilon_2 (j_s - 1)) \delta_{j-i, q_{\alpha} - q_{\beta}^{af}}.$$

4.1.3 Different Sectors

For the N node $SU(N)$ linear quiver theory on \mathbb{C}/\mathbb{Z}_p , we have different branches determined by discrete charges. The generic formula for a specific branch of orbifold instanton partition function is

$$Z_{\text{Quiver}}(a, m, \{q_{\alpha}^{(L)}\}; \{q_{\alpha}^{\text{af}}\}; \{q_{\alpha}^f\}) = \sum_Y \prod_{\beta=1}^N z_{\beta}^{|Y^{(\beta)}|} Z_{\Gamma}^{\text{antifund},q_{\beta}^{\text{af}}}(a, m, Y^{(1)}) \quad (4.1.6) \\ Z_{\Gamma}^{\text{fund},q_{\beta}^f}(a, m, Y^{(N)}) Z_N^{\Gamma}(\{q_{\alpha}^{(N)}\}, Y^{(N)}) \\ \prod_{L=1}^{N-1} Z_L^{\Gamma}(\{q_{\alpha}^{(L)}\}, Y^{(L)}) Z_{L,L+1}^{\Gamma}(Y^L, Y^{L+1}),$$

where

$$Z_L^{\Gamma}(\{q_{\alpha}^{(L)}\}, Y^{(L)}) := Z_{\Gamma}^{\text{vec}}(a^{(L)}, Y^{(L)}, \{q_{\alpha}^{(L)}\}), \\ Z_{L,L+1}^{\Gamma}(Y^L, Y^{L+1}) := Z_{\Gamma}^{\text{bifund},L}(a^{(L)}, m_L, \{q_{\alpha}^{(L)}\}, \{q_m^{(L)}\}, Y^L, Y^{L+1}),$$

and z_β is the gauge coupling of the β -th gauge factor. In general, orbifold instanton counting has two counting parameters if the first Chern class, c_1 , of orbifold instanton moduli space is nontrivial. For simplicity we will only consider the case when $c_1 = 0$.

We will see later, in order to extract vortex partition functions from that of instantons, up-to the Weyl symmetry, we need to choose the discrete charges in the following way: $q_\alpha^{(1)} - q_\alpha^f = \delta_{1,\alpha} \bmod p$ and $q_\alpha^{(L)} - q_\alpha^{(L+1)} + q_m = \delta_{\alpha,L+1} \bmod p$.

4.2 Vortices on \mathbb{C}/\mathbb{Z}_p

In the following we will studying the orbifold vortex counting problem. We have shown in chapter two the moduli space of vortices can be considered as a Lagrangian submanifold of the moduli space of instantons. Based on the results in [29], where the moduli space of orbifold vortex was analyzed using the moduli matrix method and the fact that $\Gamma = \mathbb{Z}_p$ is a subgroup of the localization $U(1)$, we can argue that similar instanton/vortex relation exists for the orbifold case. Recall that the moduli space of vortex partition function on \mathbb{C} is given by following ADHM like data,

$$\mathcal{M}_{N,k} = \{ (B, I) \mid [B, B^\dagger] + II^\dagger = c\mathbb{I}_k \} / U(k),$$

where $B \in \text{End}(V, V)$, $I \in \text{Hom}(V, W)$. V and W are complex linear spaces of dimension k and N . When there is an extra \mathbb{Z}_p action, V and W have further weight decomposition,

$$\begin{aligned} V_q &= \sum_{\alpha=1}^N \sum_{j=1}^{k_\alpha} T_{a_\alpha} T_{\hbar}^{-i+1} \delta_{q_\alpha+i-1, q}, \\ W_q &= \sum_{\alpha=1}^N T_{a_\alpha} \delta_{q_\alpha, q}, \\ \chi_\Gamma &= V^* \otimes V (T_1 - 1) + W^* \otimes V = \sum_q (V_q^* V_{q+1} - V_q^* V_q + W_q^* V_q). \end{aligned} \quad (4.2.1)$$

A short calculation shows

$$\chi_\Gamma = \sum_{\alpha, \beta=1}^N T_{a_{\alpha, \beta}} \sum_{i=1}^{k_\alpha} T_{\hbar}^{-i+1+k_\beta} \delta_{-i+1+k_\beta, q_{\alpha, \beta}}. \quad (4.2.2)$$

So the vector field contribution is

$$\left(Z_{\Gamma, \text{vortex}}^{\text{vector}}(a, \hbar; k; q)\right)^{-1} = \prod_{\alpha, \beta=1}^N \prod_{i=1}^{k_\alpha} (a_{\alpha, \beta} + \hbar(k_\beta + 1 - i)) \delta_{-i+1+k_\beta, q_{\alpha, \beta}}. \quad (4.2.3)$$

Similarly, we get contributions from matter fields in fundamental representation

$$\begin{aligned} Z_{\Gamma, \text{vortex}}^{\text{fund}, q_\beta^f}(a, m, \hbar; k) &= \prod_{\alpha=1}^N \prod_{\beta=1}^F \prod_{i=1}^{k_\alpha} (a_\alpha - m_\beta + \hbar(i - 1)) \delta_{1-i, q_\alpha - q_\beta^f}, \\ Z_{\Gamma, \text{vortex}}^{\text{antifund}, q_\beta^{af}}(a, m, \hbar; k) &= \prod_{\alpha=1}^N \prod_{\beta=1}^F \prod_{s \in Y_\alpha} (a_\alpha + m_\beta + \hbar(i - 1)) \delta_{1-i, q_\alpha - q_\beta^{af}}. \end{aligned} \quad (4.2.4)$$

Orbifold vortex partition functions also have many sectors determined by discrete charges:

$$Z_{\text{vortex}}(\{a, m, q_\alpha^{(L)}\}; \{q_\alpha^f\}) = \sum_k \prod_{\beta=1}^N z_\beta^{k_\beta} Z_{\Gamma, \text{vortex}}^{\text{fund}, q_\beta^f}(a, m, k) Z_{\Gamma, \text{vortex}}^{\text{vector}}(a, k; \{q_\alpha\}), \quad (4.2.5)$$

where z_β are N counting parameters, which are related but not identical to the counting parameters in (4.1.6).

4.3 Vortex from Instantons

Generally speaking the instanton/vortex relation has two key words: counting parameters and Young tableaux. It turns out that counting parameters of instantons will be combined to give counting parameters of vortices and two dimensional Young tableaux in instanton counting will collapse in a nice way to one dimensional Young tableaux in vortex counting. For $SU(N)$ vortex, we need to consider $SU(N)$ N-node linear quiver gauge theory. The instanton partition function of this gauge theory is characterized by N N-dimensional arrows of Young-tableaux, which is noted by $Y_\alpha^{(L)}$ in (A.1.2). Then by setting masses of antifundamental hypermultiplets and bifundamental hypermultiplets to special values, the Young-tableaux are forced to have following simple form,

$$Y_\alpha^{(L)} = \begin{cases} k_L & \alpha = L \\ \emptyset & \text{otherwise} \end{cases}. \quad (4.3.1)$$

The readers should keep in mind of the δ -functions of discrete charges which means that not all of the boxes in above Young tableaux will contribute to the partition functions. Through

direct calculation, we will show how to get this constraint naturally. Then we prove the equality between this degenerate orbifold instanton partition function and the $SU(N)$ orbifold vortex partition function. A necessary tool to achieve these goals is the following proposition.

Proposition 4.3.1. *For generic orbifold space, when $m_{\alpha,\beta}^{(L)} = 0$, $Y_\alpha^{(L)}$ should equal to $Y_\beta^{(L+1)}$ and when $m_{\alpha,\beta}^{(L)} = \epsilon_2$, $Y_\beta^{(L+1)}$ should have one more row than that of $Y_\alpha^{(L)}$. In this latter situation, if we further suppose the orbifold space is \mathbb{C}/\mathbb{Z}_2 , $Y_\alpha^{(L)}$ has M rows with lengths $k_1 \leq k_2 \leq \dots \leq k_M$ and $Y_\beta^{(L+1)}$ had $M+1$ rows with lengths $l_0 \leq l_1 \leq \dots \leq l_M$, then for $1 \leq i \leq M$ either $k_i = l_{i-1}$ or $k_i = l_i + 1$.*

One important observation is that in the self-dual limit $\epsilon_+ = 0$, the boxes contribute to orbifold instanton partition function are picked out by their relative hook length. So, upto some modifications the proof of the degeneracy phenomenon in Appendix A.2 is valid for the orbifold case and the above proposition can be proved analogously.

4.3.1 Constraint from Fundamental Hypermultiplets

From the equations (4.1.5), we know that for antifundamental hypermultiplets, if we want to get $Y_\alpha = \emptyset$, it is necessary that $a_\alpha + m_f = 0$ and the box $(1, 1)$ satisfy the δ -function, that is $q_\alpha - q_\beta^{\text{af}} = 0 \pmod p$ for some β . On the other hand, if we want to reduce Y_α to be one row, then $a_\alpha + m_f = -\epsilon_2$ and the box $(1, 2)$ should satisfy the δ -function, that is $q_\alpha - q_\beta^{\text{af}} = 1 \pmod p$ for some β . In order to satisfy (4.3.1), we should take

$$\begin{cases} a_\alpha^{(1)} + m_\alpha = -\epsilon_2 \delta_{\alpha,1} \\ q_\alpha^{(1)} - q_\alpha^{\text{af}} = \delta_{\alpha,1} \pmod p \end{cases} . \quad (4.3.2)$$

4.3.2 Constraint from Bifundamental Hypermultiplets

Using Proposition 4.3.1, It is easy to find that in order to satisfy (4.3.1), following identities should be satisfied:

$$\begin{cases} m_{\alpha,\alpha}^{(L)} = \epsilon_2 \delta_{\alpha,L+1} \\ Q_{\alpha,\alpha}^{(L)} = -\delta_{\alpha,L+1} \pmod p \end{cases} , \quad (4.3.3)$$

which means:

$$m_{\alpha,\beta}^{(L)} = \begin{cases} a_{\alpha,\beta}^{(L)} = a_{\alpha,\beta}^{(L+1)} & \alpha \in [1, L]; \beta = [1, L] \\ a_{\alpha,\beta}^{(L+1)} & \alpha \in [1, L]; \beta \in [L+1, N] \\ a_{\alpha,\beta}^{(L)} & \alpha \in [L+1, N]; \beta = [1, L] \end{cases} ,$$

and

$$Q_{\alpha,\beta}^{(L)} = \begin{cases} q_{\alpha,\beta}^{(L)} = q_{\alpha,\beta}^{(L+1)} & \alpha \in [1, L]; \beta = [1, L] \\ q_{\alpha,\beta}^{(L+1)} & \alpha \in [1, L]; \beta \in [L+1, N] \\ q_{\alpha,\beta}^{(L)} & \alpha \in [L+1, N]; \beta = [1, L] \end{cases} .$$

We see that the pattern of $Q_{\alpha,\beta}^{(L)}$ is the same as that of $m_{\alpha,\beta}^{(L)}$. This is a necessary consistent condition to extract orbifold vortex partition functions from orbifold instanton partition functions. The following subsection contains technical details of this statement.

4.3.3 Reshuffling the Partition Function

In order to make formulae lighter, we will make the δ -functions of discrete charges implicit and use following notations

$$(x)_k^+ := (x)_k = \prod_{i=0}^{k-1} (x + i\epsilon_2) , \quad (x)_k^- := \prod_{i=1}^k (x - \epsilon_2 i) .$$

Now let's input (4.3.3) into (4.1.6) and find the contribution from the L -th vector-multiplet as

$$\begin{aligned} (Z_L^\Gamma)^{-1} &= A \cdot B \cdot C , \\ A &= \prod_{\alpha,\beta=1}^L \left(a_{\alpha,\beta}^{(L)} \right)_{k_\alpha, k_\beta} , \\ B &= \prod_{\alpha=1}^L \prod_{\beta=L+1}^N (-1)^{k_\alpha} \left(a_{\beta,\alpha}^{(L)} \right)_{k_\alpha} = \prod_{\alpha=1}^L \prod_{\beta=L+1}^N \left(a_{\alpha,\beta}^{(L)} \right)_{k_\alpha}^- , \\ C &= \prod_{\beta=1}^L \prod_{\alpha=L+1}^N \left(a_{\alpha,\beta}^{(L)} \right)_{k_\beta}^+ . \end{aligned} \tag{4.3.4}$$

After suitable reshuffling we also get the contribution from the L -th bifundamental hypermultiplet as

$$\begin{aligned}
Z_{L,L+1}^\Gamma &= \text{I} \cdot \text{II} \cdot \text{III}, \\
\text{I} &= \left\{ \prod_{\alpha=1}^L \prod_{\beta=1}^L \left(m_{\alpha,\beta}^{(L)} \right)_{k_\alpha, k_\beta} \right\}, \\
\text{II} &= \left\{ \prod_{\alpha=1}^L \prod_{\beta=L+2}^N \left(m_{\alpha,\beta}^{(L)} \right)_{k_\alpha}^- \right\} \left\{ \prod_{\alpha=1}^L \left(m_{\alpha,L+1}^{(L)} \right)_{k_\alpha, k_{L+1}} \right\}, \\
\text{III} &= \left\{ \prod_{\alpha=L+1}^N \prod_{\beta=1}^L \left(m_{\alpha,\beta}^{(L)} \right)_{k_\beta}^+ \right\} \left\{ \prod_{\alpha=L+1}^N \left(m_{\alpha,L+1}^{(L)} \right)_{k_{L+1}}^+ \right\},
\end{aligned} \tag{4.3.5}$$

so,

$$Z_L^\Gamma Z_{L,L+1}^\Gamma = \frac{\left\{ \prod_{\alpha=1}^L \left(a_{\alpha,L+1}^{(L+1)} \right)_{k_\alpha, k_{L+1}} \right\}}{\left\{ \prod_{\alpha=1}^L \left(a_{\alpha,L+1}^{(L)} \right)_{k_\alpha}^- \right\}} \left\{ \prod_{\alpha=L+2}^N \left(a_{\alpha,L+1}^{(L+1)} \right)_{k_{L+1}}^+ \right\} (\epsilon_2)_{k_{L+1}}^+. \tag{4.3.6}$$

Other factors are

$$Z_\Gamma^{\text{fund}} = \prod_{f=1}^N \prod_{i=1}^{k_1} \left(a_1^{(1)} + m_f - \epsilon_2(i-1) \right) = (-\epsilon_2)_{k_1}^- \prod_{f=2}^N \left(a_{1,f}^{(1)} \right)_{k_1}^-, \tag{4.3.7}$$

$$Z_N^\Gamma = \prod_{\alpha=1}^N (\epsilon_2)_{k_\alpha}^+ (\epsilon_2)_{k_\alpha}^- \prod_{\alpha < \beta}^N \left(a_{\alpha,\beta}^{(N)} \right)_{k_\alpha, k_\beta} \left(a_{\beta,\alpha}^{(N)} \right)_{k_\beta, k_\alpha}. \tag{4.3.8}$$

Parameters in above formulae are not independent, since from the explicit form of $m_{\alpha,\alpha}^{(L)}$, we know

$$a_{\alpha,\beta}^{(L+1)} - a_{\alpha,\beta}^{(L)} = -\epsilon_2 (\delta_{\alpha,L+1} - \delta_{\beta,L+1}).$$

It follows that

$$\begin{aligned}
a_{K,L}^{(L)} &= a_{K,L}^{(N)} & L \in [2, N], K < L, \\
a_{K,L+1}^{(L)} &= a_{K,L+1}^{(K)} & L \in [K, N-1], K \in [1, N-1].
\end{aligned} \tag{4.3.9}$$

Similar relations are found for discrete charges

$$\begin{aligned}
Q_{K,L}^{(L)} &= Q_{K,L}^{(N)} & L \in [2, N], K < L, \\
Q_{K,L+1}^{(L)} &= Q_{K,L+1}^{(K)} & L \in [K, N-1], K \in [1, N-1].
\end{aligned} \tag{4.3.10}$$

This induce the identification of following factors

$$\prod_{L=1}^{N-1} \left\{ \prod_{\alpha=1}^L \left(a_{\alpha, L+1}^{(L+1)} \right)_{k_{\alpha}, k_{L+1}} \right\} = \prod_{\alpha < \beta}^N \left(a_{\beta, \alpha}^{(N)} \right)_{k_{\beta}, k_{\alpha}} \left(a_{\alpha, \beta}^{(N)} \right)_{k_{\alpha}, k_{\beta}},$$

$$\left\{ \prod_{f=2}^N \left(a_{1, f}^{(1)} \right)_{k_1} \right\} \left\{ \prod_{L=1}^{N-1} \prod_{\alpha=L+2}^N \left(a_{L+1, \alpha}^{(L+1)} \right)_{k_{L+1}} \right\} = \prod_{L=1}^{N-1} \left\{ \prod_{\alpha=1}^L \left(a_{\alpha, L+1}^{(L)} \right)_{k_{\alpha}} \right\}.$$

With these identities we have :

$$Z_{\text{Quiver}}(k) = \frac{\prod_{\beta=1}^N Z_{\Gamma}^{\text{fund}, q_{\beta}^f}(Y^{(N)})}{\prod_{\alpha=1}^N (\epsilon_2)_{k_{\alpha}} \prod_{\alpha < \beta}^N \left(a_{\beta, \alpha}^{(N)} \right)_{k_{\beta}, k_{\alpha}}}. \quad (4.3.11)$$

The equality in above formula is exact upto an overall sign factor which will disappear after redefine counting parameters. We recognize that the formula above is the same as the orbifold vortex partition function, if we identify $a_{\alpha}^{(N)}$ and ϵ_2 in (4.1.6) with a_{α} and \hbar in (4.2.5). A comment here is that the moduli space of orbifold instanton may have nontrivial first Chern class. We will concentrate on the case when the first Chern class is trivial which will give extra constraints on Young-tableaux. But this does not affect all the arguments in this section.

4.4 Vortex on \mathbb{C}/\mathbb{Z}_2 and $\mathcal{N} = 1$ Super Liouville Theory

In [42, 43, 44, 45] and [46, 47, 48, 49] people discussed about AGT like relation between instanton partition functions on $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathcal{N} = 1$ super Liouville theory. In the following, we will study the relation between $SU(2)$ vortex on \mathbb{C}/\mathbb{Z}_2 and degenerate states in $\mathcal{N} = 1$ super Liouville theory.

4.4.1 $SU(2)$ Vortex on \mathbb{C}/\mathbb{Z}_2

In order to compare orbifold vortex partition functions with conformal blocks of the $\mathcal{N} = 1$ super Liouville theory, it is convenient to rewrite vortex partition functions as linear differential operators acting on products of hypergeometric functions.

Vector Field Contribution

$$\begin{aligned}
(Z_{\Gamma, \text{vortex}}^{\text{vector}}(a, \hbar; k; q_{1,2}))^{-1} &= U_{\Gamma, \text{vortex}}^{\text{vector}}(\hbar, k) O_{\Gamma, \text{vortex}}^{\text{vector}}(a, \hbar; k; q_{1,2}), \quad (4.4.1) \\
U_{\Gamma, \text{vortex}}^{\text{vector}}(\hbar, k) &= \prod_{\alpha=1}^2 (2\hbar)^{\lfloor \frac{k_\alpha}{2} \rfloor} \left\lfloor \frac{k_\alpha}{2} \right\rfloor!, \\
O_{\Gamma, \text{vortex}}^{\text{vector}}(a, \hbar; k; q_{1,2}) &= \prod_{i=1}^{k_1} (a_{1,2} + \hbar(k_2 + 1 - i)) \delta_{-i+1+k_2, q_{1,2}} \\
&\quad \prod_{j=1}^{k_2} (a_{2,1} + \hbar(k_1 + 1 - j)) \delta_{-j+1+k_1, q_{2,1}}.
\end{aligned}$$

where $\lfloor x \rfloor$ is the floor function that is the largest integer not greater than x . The first part in above formula is an abelian factor. By abelian, we mean that it is the same as corresponding part of abelian vortex partition functions. The second part can be considered as the essential factor in nonabelian vortex theories. The contributions to the partition functions from vector fields are classified by $q_{1,2}$. Since $q_{1,2}$ takes value in \mathbb{Z}_2 , there are two different branches. In the following, we will set $a_1 = a, a_2 = -a$ and rewrite the second part as

$$O_{\Gamma, \text{vortex}}^{\text{vector}}(a, \hbar; k; 0) = D_{k_1, k_2}^0 \prod_{i=1}^{\lfloor \frac{k_1}{2} \rfloor} (-2a + 2\hbar i)(2\hbar i) \prod_{i=1}^{\lfloor \frac{k_2}{2} \rfloor} (2a + 2\hbar i)(2\hbar i), \quad (4.4.2)$$

$$O_{\Gamma, \text{vortex}}^{\text{vector}}(a, \hbar; k; 1) = D_{k_1, k_2}^1 \prod_{i=1}^{\lceil \frac{k_1}{2} \rceil} (-2a + \hbar(2i - 1)) \prod_{i=1}^{\lceil \frac{k_2}{2} \rceil} (2a + \hbar(2i - 1)), \quad (4.4.3)$$

where the pre-factors are defined as

$$D_{k_1, k_2}^0 := \begin{cases} \frac{2a(-1)^{\frac{k_1+k_2}{2}}}{2a+\hbar(k_2-k_1)}(-1)^{k_1} & k_1 + k_2 \text{ even} \\ (-1)^{\frac{k_2+1+k_1}{2}} 2a & k_1 + k_2 \text{ odd} \end{cases}, \quad (4.4.4)$$

$$D_{k_1, k_2}^1 = \begin{cases} (-1)^{\frac{k_1+k_2}{2}}(-1)^{k_1} & k_1 + k_2 \text{ even} \\ \frac{(-1)^{\frac{k_2-1+k_1}{2}}}{2a+\hbar(k_2-k_1)}(-1)^{k_1} & k_1 + k_2 \text{ odd} \end{cases}. \quad (4.4.5)$$

These pre-factors will turn out to be linear differential operators acting on orbifold vortex partition functions.

Fundamental Hypermultiplets Contribution

Since $q_\alpha - q_f$ can only take values of 0 and 1, there are four type contributions from fundamental hypermultiplets. When $q_{1,2} = 0$, we have

$$\begin{aligned} Z_{\Gamma, \text{vortex}}^{\text{fund}, 0, 0}(a, m_f, \hbar; k) &= \prod_{\alpha=1}^2 \prod_{i=1}^{\lceil \frac{k_\alpha}{2} \rceil} (m_{\alpha, f} + 2\hbar(i-1)) , \\ Z_{\Gamma, \text{vortex}}^{\text{fund}, 0, 1}(a, m, \hbar; k) &= \prod_{\alpha=1}^2 \prod_{i=1}^{\lfloor \frac{k_\alpha}{2} \rfloor} (m_{\alpha, f} + \hbar(2i-1)) , \end{aligned} \quad (4.4.6)$$

where $m_{\alpha, f} = a_\alpha - m_f$ and $\lceil x \rceil$ is the ceiling function that is the smallest integer not less than x . When $q_{1,2} = 1$, we have

$$\begin{aligned} Z_{\Gamma, \text{vortex}}^{\text{fund}, 1, 0}(a, m_f, \hbar; k) &= \prod_{i=1}^{\lceil \frac{k_1}{2} \rceil} (m_{1, f} + 2\hbar(i-1)) \prod_{i=1}^{\lfloor \frac{k_2}{2} \rfloor} (m_{2, f} + \hbar(2i-1)) , \\ Z_{\Gamma, \text{vortex}}^{\text{fund}, 1, 1}(a, m, \hbar; k) &= \prod_{i=1}^{\lfloor \frac{k_1}{2} \rfloor} (m_{1, f} + \hbar(2i-1)) \prod_{i=1}^{\lceil \frac{k_2}{2} \rceil} (m_{2, f} + 2\hbar(i-1)) . \end{aligned} \quad (4.4.7)$$

Notice that on the LHS of the formula above we use two integers in the superscript to denote the types of fundamental hypermultiplet contributions.

Vortex Partition Functions

Unlike non-orbifold case, where there is only one vortex partition function, orbifold vortex partition function has many sectors characterized by discrete charges.

$$\begin{aligned} Z_{\Gamma}^{\text{vortex}}(q_{1,2}, p_1, p_2; k) &:= Z_{\Gamma, \text{vortex}}^{\text{vector}}(a, \hbar; k; q_{1,2}) Z_{\Gamma, \text{vortex}}^{\text{fund}, q_{1,2}, p_1}(a, m_1, \hbar; k) \\ &\quad Z_{\Gamma, \text{vortex}}^{\text{fund}, q_{1,2}, p_2}(a, m_2, \hbar; k) . \end{aligned} \quad (4.4.8)$$

On the LHS of above formula we make a and the mass parameters implicit to make the formula shorter. In general there are eight different types, since the integers of the LHS can only take values in 0 and 1. Four examples related to our discussion are

$$Z_{\Gamma}^{\text{vortex}}(0, 0, 0; k) = \frac{1}{D_{k_1, k_2}^0} \frac{\prod_{\alpha=1}^2 \prod_{i=1}^{\lceil \frac{k_\alpha}{2} \rceil} (m_{\alpha, 1} + 2\hbar(i-1)) \prod_{i=1}^{\lceil \frac{k_\alpha}{2} \rceil} (m_{\alpha, 2} + 2\hbar(i-1))}{\prod_{i=1}^{\lfloor \frac{k_1}{2} \rfloor} (-2a + 2\hbar i)(2\hbar i) \prod_{i=1}^{\lfloor \frac{k_2}{2} \rfloor} (2a + 2\hbar i)(2\hbar i)} , \quad (4.4.9)$$

$$Z_{\Gamma}^{\text{vortex}}(0, 0, 1; k) = \frac{1}{D_{k_1, k_2}^0} \frac{\prod_{\alpha=1}^2 \prod_{i=1}^{\lceil \frac{k_{\alpha}}{2} \rceil} (m_{\alpha,1} + 2\hbar(i-1)) \prod_{i=1}^{\lfloor \frac{k_{\alpha}}{2} \rfloor} (m_{\alpha,2} + 2\hbar(i-1))}{\prod_{i=1}^{\lfloor \frac{k_1}{2} \rfloor} (-2a + 2\hbar i)(2\hbar i) \prod_{i=1}^{\lfloor \frac{k_2}{2} \rfloor} (2a + 2\hbar i)(2\hbar i)}, \quad (4.4.10)$$

$$Z_{\Gamma}^{\text{vortex}}(1, 0, 0; k) = \frac{1}{D_{k_1, k_2}^1} \frac{\prod_f^2 \prod_{i=1}^{\lceil \frac{k_1}{2} \rceil} (m_{1,f} + 2\hbar(i-1)) \prod_{i=1}^{\lfloor \frac{k_2}{2} \rfloor} (m_{2,f} + \hbar(2i-1))}{\prod_{\alpha=1}^2 (2\hbar)^{\lfloor \frac{k_{\alpha}}{2} \rfloor} \lfloor \frac{k_{\alpha}}{2} \rfloor! \prod_{i=1}^{\lceil \frac{k_1}{2} \rceil} (-2a + \hbar(2i-1)) \prod_{i=1}^{\lceil \frac{k_2}{2} \rceil} (2a + \hbar(2i-1))}, \quad (4.4.11)$$

$$Z_{\Gamma}^{\text{vortex}}(1, 0, 1; k) = \frac{1}{D_{k_1, k_2}^0} \frac{\prod_f^2 \prod_{i=1}^{\lfloor \frac{k_1}{2} \rfloor} (m_{1,f} + \hbar(2i-1)) \prod_{i=1}^{\lceil \frac{k_2}{2} \rceil} (m_{2,f} + 2\hbar(i-1))}{\prod_{\alpha=1}^2 (2\hbar)^{\lfloor \frac{k_{\alpha}}{2} \rfloor} \lfloor \frac{k_{\alpha}}{2} \rfloor! \prod_{i=1}^{\lceil \frac{k_1}{2} \rceil} (-2a + \hbar(2i-1)) \prod_{i=1}^{\lceil \frac{k_2}{2} \rceil} (2a + \hbar(2i-1))}. \quad (4.4.12)$$

Since there are more branches of orbifold instanton partition functions than the types of four point correlation functions, it is reasonable that not all kinds of orbifold instanton partition function has a super Liouville theory explanation. Correspondingly not all of above vortex partition functions will correspond to correlation functions with degenerate states in super Liouville theory. Considering the symmetry between fundamental and antifundamental hypermultiplets of linear quiver gauge theories, we will show in following subsections only (4.4.10), (4.4.11), and (4.4.12) may have conformal field theory explanations. Let's first concentrate on (4.4.10).

$$\begin{aligned} Z_{\Gamma}^{\text{vortex}}(0, 0, 1) &:= \sum_k z_1^{k_1} z_2^{k_2} Z_{\Gamma}^{\text{vortex}}(0, 0, 1; k) := \\ &\sum_l (z_1^{2l_1} z_2^{2l_2} Z_{\Gamma}^{\text{vortex}}(0, 0, 1; \{2l_1, 2l_2\}) + z_1^{2l_1} z_2^{2l_2+1} Z_{\Gamma}^{\text{vortex}}(0, 0, 1; \{2l_1, 2l_2+1\}) \\ &+ z_1^{2l_1+1} z_2^{2l_2} Z_{\Gamma}^{\text{vortex}}(0, 0, 1; \{2l_1+1, 2l_2\}) + z_1^{2l_1+1} z_2^{2l_2+1} Z_{\Gamma}^{\text{vortex}}(0, 0, 1; \{2l_1+1, 2l_2+1\})), \end{aligned}$$

where l_1 and l_2 are non-negative integers.

For l_1 and l_2 even,

$$\begin{aligned} \sum_l z_1^{2l_1} z_2^{2l_2} Z_{\Gamma}^{\text{vortex}}(0, 0, 1; \{2l_1, 2l_2\}) &= \\ \left(1 + \frac{\hbar}{2a} (z_2 \partial_{z_2} - z_1 \partial_{z_1})\right) F\left(\frac{m_{1,1}}{2\hbar}, \frac{m_{1,2}}{2\hbar}, \frac{-2a}{2\hbar} + 1, -z_1^2\right) F\left(\frac{m_{2,1}}{2\hbar}, \frac{m_{2,2}}{2\hbar}, \frac{2a}{2\hbar} + 1, -z_2^2\right). \end{aligned} \quad (4.4.13)$$

For l_1 even and l_2 odd,

$$\sum_l z_1^{2l_1} z_2^{2l_2+1} Z_{\Gamma}^{\text{vortex}}(0, 0, 1; \{2l_1, 2l_2 + 1\}) = \quad (4.4.14)$$

$$\frac{-z_2}{2a} F\left(\frac{m_{1,1}}{2\hbar}, \frac{m_{1,2}}{2\hbar}, \frac{-2a}{2\hbar} + 1, -z_1^2\right) F\left(\frac{m_{2,1}}{2\hbar}, \frac{m_{2,2}}{2\hbar}, \frac{2a}{2\hbar} + 1, -z_2^2\right).$$

For l_1 odd and l_2 even,

$$\sum_l z_1^{2l_1+1} z_2^{2l_2} Z_{\Gamma}^{\text{vortex}}(0, 0, 1; \{2l_1 + 1, 2l_2\}) = \quad (4.4.15)$$

$$-\frac{m_{1,1}m_{1,2}z_1}{2a} F\left(\frac{m_{1,1}}{2\hbar} + 1, \frac{m_{1,2}}{2\hbar} + 1, \frac{-2a}{2\hbar} + 1, -z_1^2\right) F\left(\frac{m_{2,1}}{2\hbar}, \frac{m_{2,2}}{2\hbar}, \frac{2a}{2\hbar} + 1, -z_2^2\right).$$

For l_1 odd and l_2 odd,

$$\sum_l z_1^{2l_1+1} z_2^{2l_2+1} Z_{\Gamma}^{\text{vortex}}(0, 0, 1; \{2l_1 + 1, 2l_2 + 1\}) = \quad (4.4.16)$$

$$z_1 z_2 m_{1,1} m_{1,2} \left(1 + \frac{\hbar}{2a} (z_2 \partial_{z_2} - z_1 \partial_{z_1})\right) F\left(\frac{m_{1,1}}{2\hbar} + 1, \frac{m_{1,2}}{2\hbar} + 1, \frac{-2a}{2\hbar} + 1, -z_1^2\right)$$

$$F\left(\frac{m_{2,1}}{2\hbar}, \frac{m_{2,2}}{2\hbar}, \frac{2a}{2\hbar} + 1, -z_2^2\right).$$

Separately, each of them can be considered as some intertwine differential operators acting on products of two hypergeometric functions. Another type of vortex partition function which we want to calculate explicitly is (4.4.12) and the result is,

$$Z_{\Gamma}^{\text{vortex}}(1, 0, 1) := \sum_k z_1^{k_1} z_2^{k_2} Z_{\Gamma}^{\text{vortex}}(q_{1,2} = 1, 0, 1; k) := \quad (4.4.17)$$

$$\sum_l \left(z_1^{2l_1} z_2^{2l_2} Z_{\Gamma}^{\text{vortex}}(1, 0, 1; \{2l_1, 2l_2\}) + z_1^{2l_1} z_2^{2l_2+1} Z_{\Gamma}^{\text{vortex}}(1, 0, 1; \{2l_1, 2l_2 + 1\}) \right.$$

$$\left. + z_1^{2l_1+1} z_2^{2l_2} Z_{\Gamma}^{\text{vortex}}(1, 0, 1; \{2l_1 + 1, 2l_2\}) + z_1^{2l_1+1} z_2^{2l_2+1} Z_{\Gamma}^{\text{vortex}}(1, 0, 1; \{2l_1 + 1, 2l_2 + 1\}) \right).$$

For l_1 even and l_2 even,

$$\sum_l z_1^{2l_1} z_2^{2l_2} Z_{\Gamma}^{\text{vortex}}(1, 0, 1; \{2l_1, 2l_2\}) =$$

$$F\left(\frac{m_{1,1}}{2\hbar}, \frac{m_{1,2}}{2\hbar}, \frac{-2a}{2\hbar} - \frac{3}{2}, -z_1^2\right) F\left(\frac{m_{2,1}}{2\hbar}, \frac{m_{2,2}}{2\hbar}, \frac{2a}{2\hbar} - \frac{3}{2}, -z_2^2\right). \quad (4.4.18)$$

For l_1 even and l_2 odd,

$$\sum_l z_1^{2l_1} z_2^{2l_2+1} Z_{\Gamma}^{\text{vortex}}(1, 0, 1; \{2l_1, 2l_2 + 1\}) = z_2 \frac{2a + 2\hbar + \hbar(z_2 \partial_{z_2} - z_1 \partial_{z_1})}{2a + \hbar} F\left(\frac{m_{1,1}}{2\hbar}, \frac{m_{1,2}}{2\hbar}, \frac{-2a}{2\hbar} - \frac{3}{2}, -z_1^2\right) F\left(\frac{m_{2,1}}{2\hbar}, \frac{m_{2,2}}{2\hbar}, \frac{2a}{2\hbar} + \frac{1}{2}, -z_1^2\right). \quad (4.4.19)$$

For l_1 odd and l_2 even,

$$\sum_l z_1^{2l_1+1} z_2^{2l_2} Z_{\Gamma}^{\text{vortex}}(1, 0, 1; \{2l_1 + 1, 2l_2\}) = z_1 m_{1,1} m_{1,2} \frac{2a - 2\hbar + \hbar(z_2 \partial_{z_2} - z_1 \partial_{z_1})}{2a - \hbar} F\left(\frac{m_{1,1}}{2\hbar} + 1, \frac{m_{1,2}}{2\hbar} + 1, \frac{-2a}{2\hbar} + \frac{3}{2}, -z_1^2\right) F\left(\frac{m_{2,1}}{2\hbar}, \frac{m_{2,2}}{2\hbar}, \frac{2a}{2\hbar} + \frac{1}{2}, -z_2^2\right). \quad (4.4.20)$$

For l_1 odd and l_2 odd,

$$\sum_l z_1^{2l_1+1} z_2^{2l_2+1} Z_{\Gamma}^{\text{vortex}}(1, 0, 1; \{2l_1 + 1, 2l_2 + 1\}) = \frac{z_1 z_2 m_{1,1} m_{1,2}}{(-2a + \hbar)(2a + \hbar)} F\left(\frac{m_{1,1}}{2\hbar} + 1, \frac{m_{1,2}}{2\hbar} + 1, \frac{-2a}{2\hbar} + \frac{3}{2}, -z_1^2\right) F\left(\frac{m_{2,1}}{2\hbar}, \frac{m_{2,2}}{2\hbar}, \frac{2a}{2\hbar} + \frac{3}{2}, -z_2^2\right) \quad (4.4.21)$$

A universal property of $SU(2)$, \mathbb{Z}_2 orbifold vortex partition functions is that they are quadratic forms of Gaussian hypergeometric functions. This is the same for non-orbifold case and one big difference is the effective counting parameter is $2\hbar$ for orbifold case while \hbar for non-orbifold case. We will see the CFT correspondences of these properties.

4.4.2 Relation to Super Liouville Theory

Since we know the relation between orbifold vortex partition function and orbifold instanton partition function, we can find the relation between orbifold and vortex through degeneration procedure on super Liouville theory side. Recall that, $SU(N)$ vortex partition functions come from $SU(N)$ quiver gauge theory with N nodes. We are now interested in $SU(2)$ gauge theory with two nodes, and therefore we have five points on a sphere. There are in principle two ways. (1) Calculate directly the correlation function between two lowest degenerate states and three non-degenerate primary states in $\mathcal{N} = 1$ super Liouville theory. (2) If we know the complete AGT relation between partition functions of $SU(2)$ instantons on $\mathbb{C}^2/\mathbb{Z}_2$ and correlation functions of $\mathcal{N} = 1$ super Liouville theory with both Ramond and NS primary fields, we get the relation between orbifold vortex and $\mathcal{N} = 1$ super Liouville theory almost for free. However, technically both ways are difficult. There are no results concerning (1) and

(2) in the literature. In the following we will use existing results to analysis the AGT dual of orbifold vortices.

Correlation Functions with Degenerate Fields

As it is clear from previous calculation, in order to extract vortex partition functions from instanton partition functions, the parameters $m_{\alpha,\alpha}^{(L)}$ should restrict to be 0 or ϵ_2 . This means on the CFT side the fusion rule is that from lowest degenerate states, i.e. those with momentum equals $\frac{-b}{2}$. It is known that the lowest degenerate states in NS- and R-sector have momentum equal to $\frac{-3b}{2}$ and $\frac{-b}{2}$ respectively. So the CFT dual of $SU(2)$ orbifold vortex should come from five point correlation functions with two lowest degenerate states in the R-sector. Possible configurations are show in Figure 4.1, where V_{α}^R and V_{α}^{NS} denote primary fields

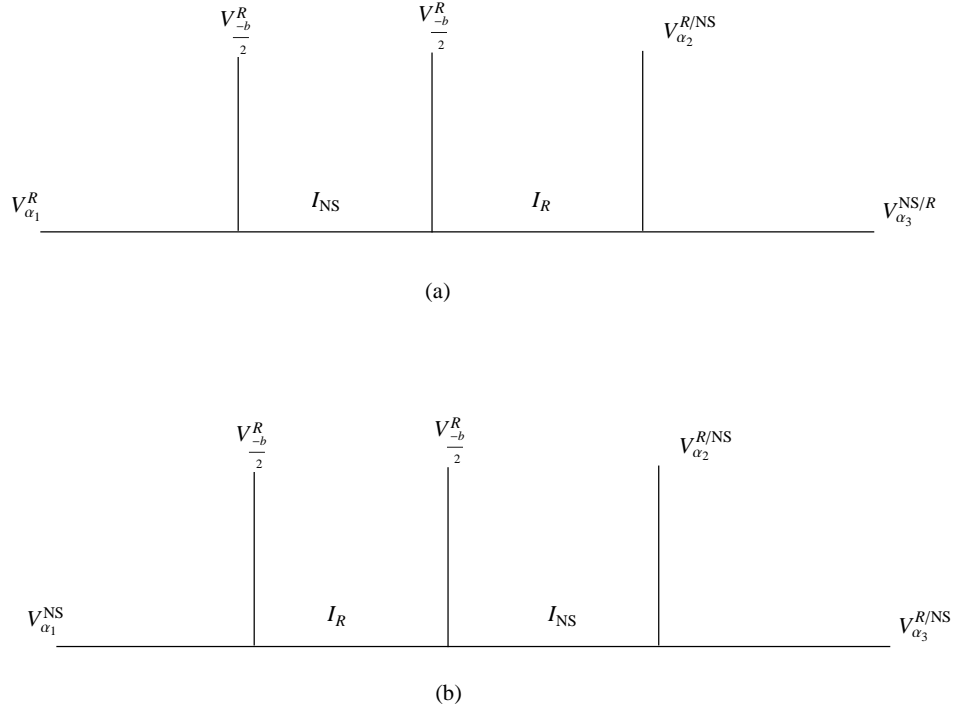


Figure 4.1: five point correlation functions corresponding to $SU(2) \mathbb{Z}_2$ orbifold vortices.

with momentum α in Ramond- and NS-sector and I_R, I_{NS} are identity operators in Ramond- and NS-sector. To exactly check our proposal, we need to know the AGT correspondence of

the following correlation functions in the super Liouville theory

$$\langle V_{\alpha_1}^R V_{\alpha_2}^R V_{\alpha_3}^R V_{\alpha_4}^R \rangle_{\text{NS}} \quad \text{and} \quad \langle V_{\alpha_1}^{\text{NS}} V_{\alpha_2}^R V_{\alpha_3}^R V_{\alpha_4}^{\text{NS}} \rangle_R, \quad (4.4.22)$$

$$\langle V_{\alpha_1}^{\text{NS}} V_{\alpha_2}^R V_{\alpha_3}^{\text{NS}} V_{\alpha_4}^R \rangle_R \quad \text{and} \quad \langle V_{\alpha_1}^R V_{\alpha_2}^R V_{\alpha_3}^{\text{NS}} V_{\alpha_4}^{\text{NS}} \rangle_{\text{NS}}. \quad (4.4.23)$$

The subscripts in above correlation functions are used to emphasize the types of internal states. Notice that except the first correlation function in (4.4.22), the other three are four point correlation function with two Ramond and two NS primary fields. The latter three are not trivially related, since they have different internal states.

The first internal state of the correlation function in Figure 4.1(a) is in NS sector and correspondingly the Kac determinant which gives denominators of conformal blocks is also in NS sector. From [42, 43, 44, 45], we can expect that $q_1^{(1)} = q_2^{(1)} \bmod 2$, since they determine the form of denominators of instanton partition functions (4.1.3). Similarly, from [50], we will conjecture that $q_1^{(2)} = q_2^{(2)} + 1 \bmod 2$. According to [31], the fusion rule of the first $V_{\frac{-b}{2}}^R$ corresponds to the choice (4.3.2), this means that when $q_1^{(1)} = q_2^{(1)} \bmod 2$, we have $q_1^{\text{af}} = q_2^{\text{af}} + 1 \bmod 2$ and when $q_1^{(1)} = q_2^{(1)} + 1 \bmod 2$, we have $q_1^{\text{af}} = q_2^{\text{af}} \bmod 2$. Our choice of the discrete charges is different from that of [42, 43, 44, 45], which in our language is $q_1^{(1)} = q_2^{(1)} \bmod 2$ and $q_1^{\text{af}} = q_2^{\text{af}} \bmod 2$. If we further consider the symmetry between fundamental and antifundamental hypermultiplets, $q_\alpha^f = q_\alpha^{\text{af}}$, we find that only (4.4.10), (4.4.11), (4.4.12) can be identified as correlation function in Figure 4.1(a).

However, presently there are no results in the literature of super Liouville theory that we can use to give a direct check of our claim. What we know are the four point correlation functions in Figure 4.2, which are calculated in [85].

For $\langle V_{\alpha_1}^{\text{NS}} V_{\frac{-b}{2}}^R V_{\alpha_2}^R V_{\alpha_3}^{\text{NS}} \rangle$, the hypergeometric function factors are

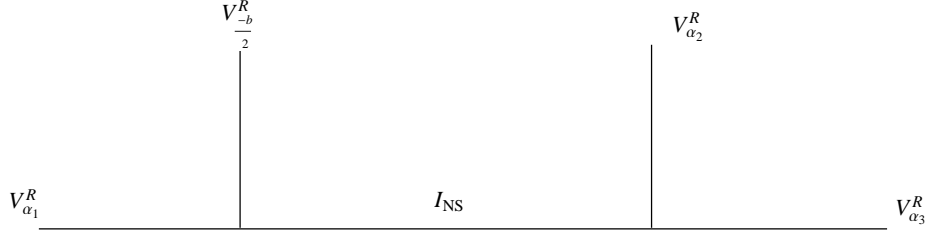
$$F\left(\frac{1}{2b^{-1}}(\alpha_1 + \alpha_3 + \alpha_4) + \frac{3}{4}, \frac{1}{2b^{-1}}(a_1 + \alpha_3 - \alpha_4) + \frac{3}{4}, \frac{2\alpha_1}{2b^{-1}} + \frac{3}{2}\right), \quad (4.4.24)$$

$$F\left(\frac{1}{2b^{-1}}(a_1 + \alpha_3 + \alpha_4) + \frac{1}{4}, \frac{1}{2b^{-1}}(a_1 + \alpha_3 - \alpha_4) + \frac{1}{4}, \frac{2\alpha_1}{2b^{-1}} + \frac{1}{2}\right). \quad (4.4.25)$$

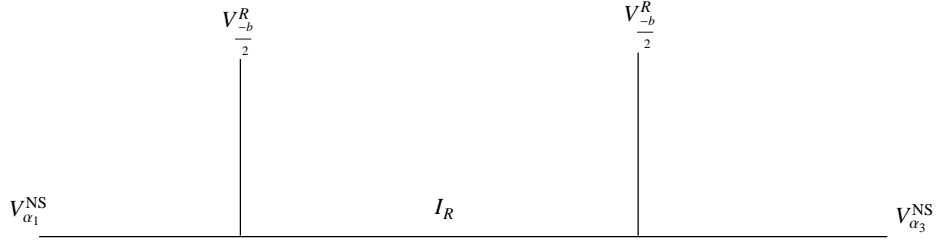
For $\langle V_{\alpha_1}^R V_{\frac{-b}{2}}^R V_{\alpha_2}^R V_{\alpha_3}^R \rangle$, the hypergeometric function factors are

$$F\left(\frac{1}{2b^{-1}}(a_1 + \alpha_3 + \alpha_4) + \frac{3}{4}, \frac{1}{2b^{-1}}(a_1 + \alpha_2 - \alpha_3) + \frac{3}{2}, \frac{2\alpha_1}{2b^{-1}} + \frac{3}{2}\right), \quad (4.4.26)$$

$$F\left(\frac{1}{2b^{-1}}(a_1 + \alpha_3 + \alpha_4) + \frac{3}{4}, \frac{1}{2b^{-1}}(a_1 + \alpha_2 - \alpha_3) + \frac{3}{2}, \frac{2\alpha_1}{2b^{-1}} + \frac{1}{2}\right). \quad (4.4.27)$$



(a)



(b)

Figure 4.2: four point correlation functions in Super Liouville theory

The one for Figure 4.2(b) is also calculated in [86] with a different convention,

$$F \left(\frac{1}{2b^{-1}} (\alpha_1 + \alpha_2 - \alpha_3) - \frac{1}{4}, \frac{1}{2b^{-1}} (\alpha_1 + \alpha_2 + \alpha_3) - \frac{1}{4}, \frac{2\alpha_1}{2b^{-1}} + 1 \right). \quad (4.4.28)$$

We can see that after a linear map between parameters of orbifold vortices and degenerate four point correlation functions in super Liouville theory, we can identify the hypergeometric function factors of both sides.

$$\begin{aligned} b^{-1} &= \hbar, \\ \alpha_1 &= a + \text{const}, \\ \alpha_2 + \alpha_3 &= m_1 + \text{const}, \\ \alpha_2 + \alpha_3 &= m_2 + \text{const}. \end{aligned}$$

The constants depends on which pair of hypergeometric functions we are comparing. This is an evidence that orbifold vortex partition functions should correspond to correlation functions of lowest degenerate Ramond fields as show in Figure 4.1. It also tells us that the identification

of parameters of orbifold instanton partition functions and that of correlation functions of the super Liouville theory in mixed sectors is the same—up to a constant shift—as in original AGT paper [7].

It is important to notice that as in non-orbifold case the four point correlation functions in Figure 4.2 can not be identified with Abelian vortex partition function, since the former has three parameters—the three momentums, while the latter has only two parameters—the two masses of fundamental hypermultiplets. So a direct check of our proposal should start from a direct clear calculation of the correlation functions in Figure 4.1, which is a hard problem due to the subtleties coming from the multi-branch of super conformal generator in R-sector and also the double vacuua in R-sector. We leave this problem in future study.

If we consider four point correlation functions with one degenerate fields as the “partition” function of surface operators, we will have two types of simple surface operators in the gauge theory dual of $\mathcal{N} = 1$ super Liouville theory, since super Liouville theory has two types of lowest degenerate states. Exactly, for \mathbb{Z}_2 orbifold $SU(2)$ gauge theory with flavor number equals 2, the instanton partition functions only have two types of lowest degeneration.

Chapter 5

Conclusions and Open Problems

In this thesis we have systematically studied properties of supersymmetric nonabelian vortices, especially various relations among vortices/simple surface operators, topological vertex, instantons and CFTs. In chapter two we calculated partition functions of nonabelian vortices using equivariant localization and found that in general vortex moduli spaces are holomorphic submanifolds of instanton moduli spaces. It follows that nonabelian vortices can be considered as interacting simple surface operators of four dimensional gauge theories. It was also found that vortex partition functions can be reduced from certain topological string amplitudes on strips. The differential/difference equations satisfied by them were checked in detail.

The instanton/vortex relation was further studied in chapter three, where an interesting degeneration phenomenon of quiver instanton partition functions was discovered and as a byproduct the exact CFT dual of nonabelian vortices was proposed. These discussions were generalized to Instanton/vortex on orbifolds and the super Liouville theory in chapter four. Conceptually, we can consider surface operators of four dimensional gauge theories, nonabelian vortices, and degenerate states of Liouville/Toda theories as images of a single object reflected by different mirrors and progress in any one of the images will lead to progress in others.

Following the results in chapter three and chapter four, it is possible that by using degeneration phenomenon on both sides of the AGT dual as a probe, one can find the complete AGT relation between gauge theories on ALE and super Liouville theories including the R-sector. Besides, one can also study surface operators of ALE gauge theories and find their CFT descriptions.

An ambitious project related with the study of this thesis is to find the five dimensional Gravity duals of two dimensional CFTs or two dimensional “holographic” descriptions of

five dimensional gravity theories. One motivation comes from the study of AdS/CFT duality [88]. One difficult and important problem in AdS/CFT is to find the gravity duals of $\mathcal{N} = 2$ four dimensional superconformal theories [89, 90]. The AGT duality may give us a bridge to overcome the difficulties of this problem. If we could find the gravity duals of two dimensional Liouville/Toda field theories, according to the AGT duality, we will obtain the gravity duals of a large set of $\mathcal{N} = 2$ four dimensional superconformal gauge theories. Since these $\mathcal{N} = 2$ four dimensional superconformal theories can be realized as the low energy effective theories of multiple M5-branes compactified on punctured Riemann surfaces, the knowledge of the gravity dual of these four dimensional theories will give us better understanding of the dynamics of the M5-branes.

A less ambitious problem is to study the feedback of the instanton/vortex relations on surface operator theories. Since present surface operator theories do not tell us much about fusion rules of degenerate fields in Liouville/Toda theories while fusion rules are of key importance in the instanton/vortex relation, it is expectable that more information about surface operators will be discovered. It should be interesting to study the physics mechanisms leading to the nice factorization formulae for quiver instanton partition functions found in chapter three and chapter four. We expect to interpret nonabelian supersymmetric vortices as specific configurations of surface operators in four dimensional gauge theories and make some progress in surface operator theories. Achievements in this direction will also feed back in the geometric Langlands program [91, 92, 93, 94] and homology knot theories [95, 96].

Actually, all the topics are related to one another and it seems that a larger network of dualities of quantum field theories in various dimensions will appear in the future.

Appendix

A.1 Instanton Partition Functions

Let us consider the instanton partition function of a linear quiver with N nodes. The corresponding brane construction has $N + 2$ sets of D4-branes and $N + 1$ NS5 branes. The complete instanton partition functions should sum over all possible Young tableaux. For a given Young tableaux figuration, the instanton partition function has following form:

$$Z_{\text{Quiver}} = Z_{\text{fund}} Z_{\text{antifund}} Z_N \prod_{i=1}^{N-1} Z_i Z_{i,i+1} . \quad (\text{A.1.1})$$

Z_{fund} and Z_{antifund} are the contributions from fundamental and antifundamental hypermultiplets respectively. Z_i is the contribution from the i -th gauge factor, while $Z_{i,i+1}$ is the contribution from the i -th bifundamental hyper. These depend on the following parameters

$$\{Y_{\alpha}^{(L)}\}_{\alpha=1}^N \quad : \quad \text{the Young tableaux of the } L\text{-th gauge factor.} \quad (\text{A.1.2})$$

$$\begin{aligned} \{a_{\alpha}^{(L)}\}_{\alpha=1}^N & : \quad \text{the Coulomb branch parameters of the } L\text{-th gauge factor.} \\ m_i & = \quad \text{the } i\text{-th mass of bifundamental hypermultiplet} \\ \mu_i & = \begin{cases} \text{masses of antifundamental hypermultiplets} & i \in [1, N] \\ \text{masses of fundamental hypermultiplets} & i \in [N + 1, 2N] \end{cases} \\ m_{\alpha,\beta}^{(L)} & := a_{\alpha}^{(L)} - a_{\beta}^{(L+1)} - m_L \end{aligned} \quad (\text{A.1.3})$$

More explicitly, for fundamental and antifundamental matter fields,

$$Z_{\text{antifund}}(a^{(1)}, \mu, Y^{(1)}) = \prod_{f=1}^N \prod_{\alpha=1}^N \prod_{(i,j) \in Y_{\alpha}^{(1)}} (\alpha_{\alpha}^{(1)} + \mu_f + \epsilon_1(i-1) + \epsilon_2(j-1)) ,$$

$$Z_{\text{fund}}(a^{(N)}, \mu, Y^{(N)}) = \prod_{f=1}^N \prod_{\alpha=1}^N \prod_{(i,j) \in Y_{\alpha}^{(N)}} (\alpha_{\alpha}^{(N)} - \mu_{N+f} + \epsilon_1 i + \epsilon_2 j) .$$

For L -th bifundamental hypermultiplet,

$$Z_{L,L+1} = \prod_{\alpha=1}^N \prod_{\beta=1}^N Z_{L,L+1}^{(\alpha,\beta)} , \quad (\text{A.1.4})$$

$$Z_{L,L+1}^{(\alpha,\beta)} = \prod_{s \in Y_{\alpha}^{(L)}} \left(m_{\alpha,\beta}^{(L)} - \epsilon_1 L_{Y_{\beta}^{(L+1)}}(s) + \epsilon_2 \left(A_{Y_{\alpha}^{(L)}}(s) + 1 \right) \right) \\ \prod_{t \in Y_{\beta}^{(L+1)}} \left(m_{\alpha,\beta}^{(L)} + \epsilon_1 \left(L_{Y_{\alpha}^{(L)}}(t) + 1 \right) - \epsilon_2 A_{Y_{\beta}^{(L+1)}}(t) \right) ,$$

$$m_{\alpha,\beta}^{(L)} := a_{\alpha}^{(L)} - a_{\beta}^{(L+1)} - m_L . \quad (\text{A.1.5})$$

$$(\text{A.1.6})$$

For the L -th gauge factor,

$$Z_L = \prod_{\alpha=1}^N \prod_{\beta=1}^N Z_L^{(\alpha,\beta)} , \quad (\text{A.1.7})$$

$$\left(Z_L^{(\alpha,\beta)} \right)^{-1} = \prod_{s \in Y_{\alpha}^{(L)}} \left(a_{\alpha,\beta}^{(L)} - \epsilon_1 L_{Y_{\beta}^{(L+1)}}(s) + \epsilon_2 \left(A_{Y_{\alpha}^{(L)}}(s) + 1 \right) \right) \\ \prod_{t \in Y_{\beta}^{(L+1)}} \left(a_{\alpha,\beta}^{(L)} + \epsilon_1 \left(L_{Y_{\alpha}^{(L)}}(t) + 1 \right) - \epsilon_2 A_{Y_{\beta}^{(L+1)}}(t) \right) .$$

For the study of vortices, we will focus on unrefined limit $\epsilon_1 = -\epsilon_2$. Formulae are simplified,

$$Z_{\text{antifund}}(a^{(1)}, \mu, Y^{(1)}) = \prod_{f=1}^N \prod_{\alpha=1}^N \prod_{(i,j) \in Y_{\alpha}^{(1)}} (a_{\alpha}^{(1)} + \mu_f + \epsilon_2(j-i)) , \quad (\text{A.1.8})$$

$$Z_{\text{fund}}(a^{(N)}, \mu, Y^{(N)}) = \prod_{f=1}^N \prod_{\alpha=1}^N \prod_{(i,j) \in Y_{\alpha}^{(N)}} (a_{\alpha}^{(N)} - \mu_{f+N} + \epsilon_2(j-i)) . \quad (\text{A.1.9})$$

The L -th bifundamental hypermultiplet contribution is,

$$Z_{L,L+1}^{(\alpha,\beta)} = \prod_{s \in Y_\alpha^{(L)}} \left(m_{\alpha,\beta}^{(L)} + \epsilon_2 \left(A_{Y_\alpha^{(L)}}(s) + L_{Y_\beta^{(L+1)}}(s) + 1 \right) \right) \prod_{t \in Y_\beta^{(L+1)}} \left(m_{\alpha,\beta}^{(L)} - \epsilon_2 \left(A_{Y_\beta^{(L+1)}}(t) + L_{Y_\alpha^{(L)}}(t) + 1 \right) \right). \quad (\text{A.1.10})$$

The L -th gauge factor contribution is,

$$\left(Z_L^{(\alpha,\beta)} \right)^{-1} = \prod_{s \in Y_\alpha^{(L)}} \left(a_{\alpha,\beta}^{(L)} + \epsilon_2 \left(A_{Y_\alpha^{(L)}}(s) + L_{Y_\beta^{(L)}}(s) + 1 \right) \right) \prod_{t \in Y_\beta^{(L)}} \left(a_{\alpha,\beta}^{(L)} - \epsilon_2 \left(A_{Y_\beta^{(L)}}(t) + L_{Y_\alpha^{(L)}}(t) + 1 \right) \right). \quad (\text{A.1.11})$$

For a Young-tableau Y , one box s has coordinates (i, j) , where i counts the number of columns and j counts the number of rows. Then the arm and leg of s relative to another Young-tableau W , are defined as $A_W(s) := W_i - j$; $L_W(s) := W_j^t - i$. Where W^t is the dual partition of W . $|Y| := \sum_i Y_i$. We call a partition of the form (1^k) a row partition of length k , and a partition of the form (k) a column partition.

A.2 Degeneration from Bifundamental Masses

Let us state our results and then prove them. The claim is that when $m_{\alpha,\beta} = 0$, $W_\beta = Y_\alpha$ and when $m_{\alpha,\beta} = \epsilon_2$, W_β has one row more than that of Y_α . In this situation, if we suppose Y_α has L rows with lengths $k_1 \leq k_2 \leq \dots \leq k_L$ and W_β had $L + 1$ rows with lengths $l_0 \leq l_1 \leq \dots \leq l_L$, then for $1 \leq i \leq L$ either $k_i = l_{i-1}$ or $k_i = l_i + 1$. Please refer to Figure A.1 for a pictorial illustration. Let's start from the simpler case $m_{\alpha,\beta} = 0$.

$$Z_{\text{bifund}}^{(\alpha,\beta)} = \prod_{s \in Y_\alpha} \left(\epsilon_2 \left(A_{Y_\alpha}(s) + L_{W_\beta}(s) + 1 \right) \right) \prod_{t \in W_\beta} \left(-\epsilon_2 \left(A_{W_\beta}(t) + L_{Y_\alpha}(t) + 1 \right) \right).$$

Let's suppose $Y_\alpha^t = (k_1, k_2, \dots, k_L)$; $W_\beta^t = (l_1, l_2, \dots, l_M)$. We will proceed in our proof by induction from the top row to the bottom.

If $M > L$, then the result is non vanishing only if $L_{Y_\alpha}(t) + 1 = L_\emptyset(t) + 1 = -i + 1 \neq 0$. The same argument applies for $L \leq M$, so that we stay with $M = L$.

The first induction step is when t is on the top row of W_β , so that $A_{W_\beta}(t) = 0$, and

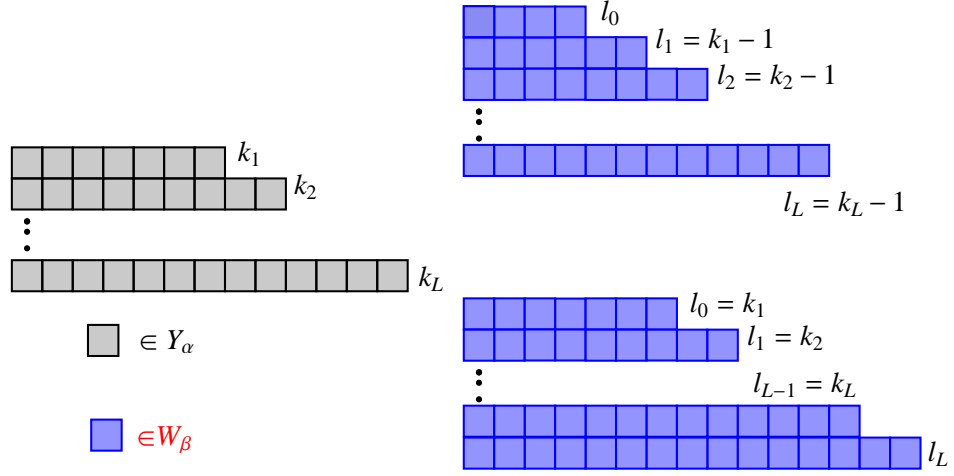


Figure A.1: Relation between Y_α and W_β when $m_{\alpha,\beta} = \epsilon_2$

Then consider contribution from s.

$L_{Y_\alpha}(t) + 1 = 1 + k_1 - i_t$, then

$$\begin{cases} L_{Y_\alpha}(t) + 1 = 1 + k_1 - i_t \neq 0 \\ i_t \in [1, l_1] \end{cases} \implies k_1 \geq l_1.$$

Similarly for the contribution from $s \in Y_\alpha$, we get $l_1 \geq k_1$, implying $k_1 = l_1$. Suppose now $k_i = l_i$ when $i \leq p - 1$ and let's prove that $k_p = l_p$.

1. $i_t \in [1, l_1]$, $A_{W_\beta}(t) = p - 1$,

$$\begin{cases} L_{Y_\alpha}(t) + p = p + k_p - i_t \neq 0 \\ i_t \in [1, l_1] \end{cases} \implies k_p \geq l_1 - (p - 1);$$

2. when $i_t \in [l_1 + 1, l_2]$, $A_{W_\beta}(t) = p - 2$,

$$\begin{cases} L_{Y_\alpha}(t) + 1 + p - 2 = p - 1 + k_p - i_t \neq 0 \\ i_t \in [1 + l_1, l_2] \end{cases} \implies k_p \leq l_1 + 1 - p \text{ or } k_p \geq l_2 - p + 2.$$

Since $k_1 = l_1$, then $k_p \neq l_1 + 1 - p$ and $k_p \geq l_2 - p + 2$. By iterating this procedure we find $k_p \geq l_p$, and symmetrically $l_p \geq k_p$, namely $l_p = k_p$. This ends the proof of the first statement.

Now let us turn to the case $m_{\alpha,\beta} = \epsilon_2$,

$$Z_{\text{bifund}}^{(\alpha,\beta)} = \prod_{s \in Y_\alpha} (\epsilon_2 (A_{Y_\alpha}(s) + L_{W_\beta}(s) + 2)) \prod_{t \in W_\beta} (-\epsilon_2 (A_{W_\beta}(t) + L_{Y_\alpha}(t))) .$$

It is easy to show that W_β can have at most one row more than Y_α . Suppose that $Y_\alpha^t = (k_1, k_2, \dots, k_L)$; $W_\beta^t = (l_0, l_1, l_2, \dots, l_M)$ and apply induction again from top to bottom.

When t is on the top row of W_β there is no constraint for the length l_0 . When t is on the next to top row of W_β then

1. for $i_t \in [1, l_0]$, in this case $A_{W_\beta}(t) = 1$,

$$\begin{cases} L_{Y_\alpha}(t) + 1 = 1 + k_1 - i_t \neq 0 \\ i_t \in [1, l_0] \end{cases} \implies k_1 \geq l_0 ;$$

2. for $i_t \in [l_0 + 1, l_1]$, in this case $A_{W_\beta}(t) = 0$,

$$\begin{cases} L_{Y_\alpha}(t) = k_1 - i_t \neq 0 \\ i_t \in [1 + l_0, l_1] \end{cases} \implies k_1 \leq l_0 \text{ or } k_1 \geq l_1 + 1 ;$$

so we have $k_1 = l_0$ or $k_1 \geq l_1 + 1$.

Let us consider now the contribution from Y_α . When s is on the top row of Y_α , $A_{Y_\alpha}(s) = 0$ and we get

$$\begin{cases} L_{W_\beta}(s) + 2 = l_1 - i_s + 2 \neq 0 \\ i_s \in [1, k_1] \end{cases} \implies l_1 \geq k_1 - 1 ,$$

so $k_1 = l_0$ or $k_1 \geq l_1 + 1 \cap l_1 \geq k_1 - 1 \implies k_1 = l_0$ or $k_1 = l_1 + 1$. Now suppose that for $i \leq p - 1$, we have $l_i = k_i - 1$ or $l_{i-1} = k_i$. Then

1. for $i_t \in [1, l_0]$, $A_{W_\beta}(t) = p$ and

$$\begin{cases} L_{Y_\alpha}(t) + p = p + k_p - i_t \neq 0 \\ i_t \in [1, l_0] \end{cases} \implies k_p \geq l_0 - p + 1 ;$$

2. for $i_t \in [l_0 + 1, l_1]$, $A_{W_\beta}(t) = p - 1$,

$$\begin{cases} L_{Y_\alpha}(t) + p - 1 = p - 1 + k_p - i_t \neq 0 \\ i_t \in [1 + l_0, l_1] \end{cases} \implies k_p \leq l_0 + 1 - p \text{ or } k_p \geq l_1 + 2 - p ;$$

so we have $k_p = l_0 + 1 - p$ or $k_p \geq l_1 + 2 - p$. By iterating this procedure we get $k_p = l_0 + 1 - p$ or $k_p = l_1 + 2 - p, \dots$, or $k_p = l_{p-1}$ or $k_p \geq l_p + 1$. From the induction assumption

we have $k_p \geq l_p + 1$ or $k_p = l_{p-1}$.

Let us now consider the contribution from $s \in Y_\alpha$,

1. for $i_s \in [1, k_1]$, $A_{Y_\alpha}(s) = p - 1$,

$$\begin{cases} L_{W_\beta}(s) + 1 + p = l_p - i_s + 1 + p \neq 0 \\ i_s \in [1, k_1] \end{cases} \implies l_p \geq k_1 - p;$$

2. for $i_s \in [k_1 + 1, k_2]$, $A_{Y_\alpha}(s) = p - 2$,

$$\begin{cases} L_{W_\beta}(s) + p = l_p - i_s + p \neq 0 \\ i_s \in [k_1 + 1, k_2] \end{cases} \implies l_p \leq k_1 - p \text{ or } l_p \geq k_2 - p + 1;$$

so we find $l_p = k_1 - p$ or $l_p \geq k_2 - p + 1$. By iterating the procedure we find $l_p = k_1 - p$ or $l_p = k_2 - p + 1, \dots$, or $l_p = k_{p-1} - 2$ or $l_p \geq k_p - 1$. From the induction assumption it follows that $l_p \geq k_p - 1$.

Finally, combining the results from W_β and Y_α , we have : $k_p = l_p + 1$ or $k_p = l_{p-1}$, which is what we wanted to prove.

Factorization Formulae

When $Y_L^t = (l_1 + 1, l_2 + 1, \dots, 1 + l_L)$ $Y_{L+1}^t = (l_0, l_1, l_2, \dots, l_L)$ ($l_i \leq l_{i+1}$),

$$\begin{aligned} (\epsilon_2)_{Y_L, Y_{L+1}} &= \prod_{(i,j) \in Y_L} (\epsilon_2 + \epsilon_2 (A_{Y_L}(i, j) + L_{Y_{L+1}}(i, j) + 1)) \\ &\quad \prod_{(a,b) \in Y_{L+1}} (\epsilon_2 - \epsilon_2 (A_{Y_{L+1}}(a, b) + L_{Y_L}(a, b) + 1)) \\ &= \prod_{(i,j) \in Y_L} (\epsilon_2 + \epsilon_2 (A_{Y_L}(i, j) + L_{Y_L}(i, j))) \\ &\quad \prod_{(a,b) \in Y_{L+1} \cap Y_L} (\epsilon_2 - \epsilon_2 (A_{Y_{L+1}}(a, b) + L_{Y_{L+1}}(a, b) + 2)) \\ &\quad \prod_{(a,b) \in Y_{L+1} \setminus Y_L} (\epsilon_2 - \epsilon_2 (A_{Y_{L+1}}(a, b) + L_{Y_L}(a, b) + 1)) \\ &= H_{Y_L} \prod_{(a,b) \in Y_{L+1} \cap Y_L} -h_{Y_{L+1}}(a, b) \prod_{(a,b) \in Y_{L+1} \setminus Y_L} h_{Y_{L+1}}(a, b) \\ &= (-1)^{|Y_L| - L} H_{Y_L} H_{Y_{L+1}}. \end{aligned}$$

Similarly, when $Y_L^t = (l_1, l_2, \dots, l_L)$ and $Y_{L+1}^t = (l_1, l_2, \dots, l_{L+1})$ ($l_i \leq l_{i+1}$),

$$\begin{aligned}
(\epsilon_2)_{Y_L, Y_{L+1}} &= \prod_{(i,j) \in Y_L} (\epsilon_2 ((A_{Y_L}(i,j) + 1) + L_{Y_{L+1}}(i,j) + 1)) \\
&\quad \prod_{(a,b) \in Y_{L+1}} (-\epsilon_2 ((A_{Y_{L+1}}(a,b) - 1) + L_{Y_L}(a,b) + 1)) \\
&= \prod_{(i,j) \in Y_L} h_{Y_{L+1}}(i,j) \prod_{(a,b) \in Y_L} (-1)^{|Y_L|} h_{Y_L}(a,b) \\
&\quad \prod_{(a,b) \in Y_{L+1} \setminus Y_L} (-\epsilon_2 ((A_{Y_{L+1}}(a,b) - 1) + L_{Y_L}(a,b) + 1)) \\
&= (-1)^{|Y_L|} H_{Y_L} H_{Y_{L+1}}.
\end{aligned}$$

A.3 Conventions on Topological Vertex

In this Appendix we summarize the usual conventions on the topological vertex on the strip and some useful formulae that we used in the main text. The normalized amplitude on the strip is [39],

$$\begin{aligned}
A_{\{\beta\}}^{\{\alpha\}} &= \prod_{a=1}^N s_{\alpha_a} s_{\beta_a} \prod_{i=-\infty}^{\infty} \prod_{a \leq b} (1 - q^i Q_{\alpha_a \beta_b})^{C_i(\alpha_a, \beta_b)} \prod_{a < b} (1 - q^i Q_{\beta_a \alpha_b})^{C_i(\beta_a^t, \alpha_b^t)} \\
&\quad \left(\prod_{a < b} (1 - q^i Q_{\alpha_a \alpha_b})^{C_i(\alpha_a, \alpha_b^t)} (1 - q^i Q_{\beta_a \beta_b})^{C_i(\beta_a^t, \beta_b)} \right)^{-1}, \tag{A.3.1}
\end{aligned}$$

where α_a, β_b are the left and right partitions parametrizing the toric branes boundary conditions. s_α is the Schur function

$$s_\alpha(q) = q^{\sum_i (i-1)\alpha_i} \prod_{p \in \alpha} \frac{1}{1 - q^{\text{hook}(p)}}, \tag{A.3.2}$$

where α_i is the i -th component of the partition α , and $\text{hook}(p)$ is the hook length of a point $p \in \alpha$ seen as a Young tableaux.

For columns and strips one has

$$\begin{aligned} s_{(1^k)} &= \prod_{i=1}^k \frac{1}{1-q^i}, \\ s_{(k)} &= q^{\frac{k(k-1)}{2}} \prod_{i=1}^k \frac{1}{1-q^i}. \end{aligned} \quad (\text{A.3.3})$$

The coefficients $C_k(\alpha, \beta)$ are defined for two given partitions α and β by the formula

$$\begin{aligned} \sum_k C_k(\alpha, \beta) q^k &= \frac{q}{(1-q)^2} \left(1 + (q-1)^2 \sum_{i=1}^{d_\alpha} q^{-i} \sum_{j=0}^{\alpha_i-1} q^j \right) \\ &\quad \left(1 + (q-1)^2 \sum_{i=1}^{d_\beta} q^{-i} \sum_{j=0}^{\beta_i-1} q^j \right) - \frac{q}{(1-q)^2}, \end{aligned} \quad (\text{A.3.4})$$

and are symmetric by definitions, that is $C_i(\alpha, \beta) = C_i(\beta, \alpha)$.

Specializing to columns and strips one finds

$$\begin{aligned} C_i(1^k, \emptyset) &= \begin{cases} 1 & i \in [0, k-1] \\ 0 & \text{otherwise} \end{cases}, \\ C_i((k), \emptyset) &= \begin{cases} 1 & i \in [-k+1, 0] \\ 0 & \text{otherwise} \end{cases}, \\ C_i(1^{k_1}, (k_2)) &= \begin{cases} 1 & i \in [-k_2, k_1 - k_2 - 1] \cup [k_1 - k_2 + 1, k_1] \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

Bibliography

- [1] N. Seiberg and E. Witten, *Monopole Condensation, And Confinement In $N=2$ Supersymmetric Yang-Mills Theory*, *Nucl.Phys.B* **426:19-52,1994**;
(*Nucl.Phys.B*426:19-52,1994; Erratum-*ibid.B*430:485-486,1994)
Erratum-*ibid.B*430:485-486,1994, [[hep-th/9407087](#)].
- [2] N. Seiberg and E. Witten, *Monopoles, Duality and Chiral Symmetry Breaking in $N=2$ Supersymmetric QCD*, *Nucl.Phys.B* **431:484-550,1994**
(*Nucl.Phys.B*431:484-550,1994) [[hep-th/9408099](#)].
- [3] E. Witten, *Solutions Of Four-Dimensional Field Theories Via M Theory*, *Nucl.Phys.B* **500:3-42,1997** (*Nucl.Phys.B*500:3-42,1997) [[hep-th/9703166](#)].
- [4] N. A. Nekrasov, *Seiberg-Witten Prepotential From Instanton Counting*,
Adv.Theor.Math.Phys. **7:831-864,2004** (*Adv.Theor.Math.Phys.*7:831-864,2004)
[[hep-th/0206161](#)].
- [5] N. Nekrasov and A. Okounkov, *Seiberg-Witten Theory and Random Partitions*,
[hep-th/0306238](#).
- [6] D. Gaiotto, *$N=2$ dualities*, [0904.2715](#).
- [7] L. F. Alday, D. Gaiotto, and Y. Tachikawa, *Liouville Correlation Functions from Four-dimensional Gauge Theories*, *Lett.Math.Phys.* **91:167-197,2010** (June, 2009)
[[0906.3219](#)].
- [8] L. F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa, and H. Verlinde, *Loop and surface operators in $N=2$ gauge theory and Liouville modular geometry*, *JHEP* **1001:113,2010** (Sept., 2009) [[0909.0945](#)].
- [9] C. Kozcaz, S. Pasquetti, and N. Wyllard, *A & B model approaches to surface operators and Toda theories*, [1004.2025](#).

- [10] T. Dimofte, S. Gukov, and L. Hollands, *Vortex Counting and Lagrangian 3-manifolds*, 1006.0977.
- [11] G. Bonelli, A. Tanzini, and J. Zhao, *Vertices, Vortices & Interacting Surface Operators*, 1102.0184.
- [12] N. Dorey, *The BPS Spectra of Two-Dimensional Supersymmetric Gauge Theories with Twisted Mass Terms*, *JHEP* **9811:005,1998** (JHEP 9811:005,1998) [hep-th/9806056].
- [13] A. Hanany and D. Tong, *Vortices, Instantons and Branes*, *JHEP* **0307** (2003) 037, [hep-th/0306150].
- [14] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi, and A. Yung, *Nonabelian Superconductors: Vortices and Confinement in $\mathcal{N} = 2$ SQCD*, *Nucl.Phys.B* **673:187-216,2003** (Nucl.Phys.B673:187-216,2003) [hep-th/0307287].
- [15] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, *Moduli Space of Non-Abelian Vortices*, *Phys.Rev.Lett.* **96:161601,2006** (Phys.Rev.Lett.96:161601,2006) [hep-th/0511088].
- [16] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, *Solitons in the Higgs phase – the moduli matrix approach*, *J.Phys.A* **39:R315-R392,2006** (J.Phys.A39:R315-R392,2006) [hep-th/0602170].
- [17] T. Fujimori, G. Marmorini, M. Nitta, K. Ohashi, and N. Sakai, *The Moduli Space Metric for Well-Separated Non-Abelian Vortices*, *Phys.Rev.D* **82:065005,2010** (Feb., 2010) [1002.4580].
- [18] M. Eto, T. Fujimori, M. Nitta, K. Ohashi, and N. Sakai, *Dynamics of Non-Abelian Vortices*, *Phys.Rev.D* **84:125030,2011** (May, 2011) [1105.1547].
- [19] Y. Yoshida, *Localization of Vortex Partition Functions in $\mathcal{N} = (2, 2)$ Super Yang-Mills theory*, 1101.0872.
- [20] A. Miyake, K. Ohta, and N. Sakai, *Volume of Moduli Space of Vortex Equations and Localization*, *Prog. Theor. Phys.* **126** (May, 2011) ,637–680, [1105.2087].
- [21] M. Eto, T. Fujimori, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta, and N. Sakai, *Non-Abelian Vortices on Cylinder – Duality between vortices and walls*, *Phys.Rev.D* **73:085008,2006** (Phys.Rev.D73:085008,2006) [hep-th/0601181].

- [22] G. S. Lozano, D. Marques, and F. A. Schaposnik, *Non-Abelian Vortices on the Torus*, *JHEP* **0709:095,2007** (Aug., 2007) [0708.2386].
- [23] A. D. Popov, *Non-Abelian Vortices on Riemann Surfaces: an Integrable Case*, *Lett.Math.Phys.* **84:139-148,2008** (Jan., 2008) [0801.0808].
- [24] J. M. Baptista, *Non-abelian vortices on compact Riemann surfaces*, *Commun.Math.Phys.* **291:799-812,2009** (Oct., 2008) [0810.3220].
- [25] J. M. Baptista, *On the L^2 -metric of vortex moduli spaces*, *Nucl.Phys.B* **844:308-333,2011** (Mar., 2010) [1003.1296].
- [26] N. S. Manton and N. A. Rink, *Vortices on Hyperbolic Surfaces*, *J. Phys. A* **43** (Dec., 2009) 434024,2010, [0912.2058].
- [27] N. S. Manton and N. A. Rink, *Geometry and Energy of Non-abelian Vortices*, *J. Math. Phys.* **52**, (Dec., 2010) 043511, [1012.3014].
- [28] N. S. Manton and N. Sakai, *Maximally Non-Abelian Vortices from Self-dual Yang–Mills Fields*, *Phys.Lett.B* **687:395-399,2010** (Jan., 2010) [1001.5236].
- [29] T. Kimura and M. Nitta, *Vortices on Orbifolds*, *JHEP* **1109:118,2011** (Aug., 2011) [1108.3563].
- [30] S. Shadchin, *On F-term contribution to effective action*, *JHEP* **0708:052,2007** (JHEP 0708:052,2007) [hep-th/0611278].
- [31] G. Bonelli, A. Tanzini, and J. Zhao, *The Liouville side of the Vortex*, 1107.2787.
- [32] N. Doroud, J. Gomis, B. L. Floch, and S. Lee, *Exact Results in $D=2$ Supersymmetric Gauge Theories*, 1206.2606.
- [33] F. Benini and S. Cremonesi, *Partition functions of $N=(2,2)$ gauge theories on S^2 and vortices*, 1206.2356.
- [34] R. Auzzi, S. Bolognesi, and J. Evslin, *Monopoles Can be Confined by 0, 1 or 2 Vortices*, *JHEP* **0502:046,2005** (JHEP 0502:046,2005) [hep-th/0411074].
- [35] S. Katz, A. Klemm, and C. Vafa, *Geometric Engineering of Quantum Field Theories*, *Nucl.Phys. B* **497** (1997) 173–195, [hep-th/9609239].
- [36] S. Katz, P. Mayr, and C. Vafa, *Mirror symmetry and Exact Solution of 4D $N=2$ Gauge Theories I*, *Adv.Theor.Math.Phys.* **1:53-114,1998** (Adv.Theor.Math.Phys.1:53-114,1998) [hep-th/9706110].

- [37] A. Karch, D. Lust, and D. J. Smith, *Equivalence of Geometric Engineering and Hanany-Witten via Fractional Branes*, *Nucl.Phys.B* **533:348-372,1998** (Nucl.Phys.B533:348-372,1998) [hep-th/9803232].
- [38] M. Aganagic, A. Klemm, M. Marino, and C. Vafa, *The Topological Vertex*, *Commun.Math.Phys.* **254** (2005) 425–478, [hep-th/0305132].
- [39] A. Iqbal and A.-K. Kashani-Poor, *The Vertex on a Strip*, *Adv.Theor.Math.Phys.* **10** (2006) 317–343, [hep-th/0410174].
- [40] A. Iqbal, C. Kozcaz, and C. Vafa, *The Refined Topological Vertex*, *JHEP* **0910:069,2009** (JHEP 0910:069,2009) [hep-th/0701156].
- [41] M. Taki, *Refined Topological Vertex and Instanton Counting*, *JHEP* **0803:048,2008** (Oct., 2007) [0710.1776].
- [42] V. Belavin and B. Feigin, *Super Liouville conformal blocks from $N=2$ $SU(2)$ quiver gauge theories*, 1105.5800.
- [43] A. Belavin, V. Belavin, and M. Bershtein, *Instantons and 2d Superconformal field theory*, *JHEP* **1109:117,2011** (June, 2011) [1106.4001].
- [44] G. Bonelli, K. Maruyoshi, and A. Tanzini, *Instantons on ALE spaces and Super Liouville Conformal Field Theories*, 1106.2505.
- [45] G. Bonelli, K. Maruyoshi, and A. Tanzini, *Gauge Theories on ALE Space and Super Liouville Correlation Functions*, 1107.4609.
- [46] T. Nishioka and Y. Tachikawa, *Para-Liouville/Toda central charges from M5-branes*, *Phys.Rev.D* **84:046009,2011** (June, 2011) [1106.1172].
- [47] N. Wyllard, *Coset conformal blocks and $N=2$ gauge theories*, 1109.4264.
- [48] M. N. Alfimov and G. M. Tarnopolsky, *Parafermionic Liouville field theory and instantons on ALE spaces*, 1110.5628.
- [49] A. A. Belavin, M. A. Bershtein, B. L. Feigin, A. V. Litvinov, and G. M. Tarnopolsky, *Instanton moduli spaces and bases in coset conformal field theory*, 1111.2803.
- [50] Y. Ito, *Ramond sector of super Liouville theory from instantons on an ALE space*, 1110.2176.
- [51] J. Zhao, *Orbifold Vortex and Super Liouville Theory*, 1111.7095.

- [52] H. Liu, J. Yang, and J. Zhao, *Matrix Model and Refined Wall-Crossing Formula*, 1011.0947.
- [53] P. Sulkowski, *Refined matrix models from BPS counting*, *Phys.Rev.D* **83:085021,2011** (Dec., 2010) [1012.3228].
- [54] D. Gaiotto, G. W. Moore, and A. Neitzke, *Wall-crossing, Hitchin Systems, and the WKB Approximation*, 0907.3987.
- [55] D. Gaiotto, G. W. Moore, and A. Neitzke, *Spectral networks*, 1204.4824.
- [56] P. C. Argyres and N. Seiberg, *S-duality in $N=2$ supersymmetric gauge theories*, *JHEP* **0712:088,2007** (Nov., 2007) [0711.0054].
- [57] F. Fucito, J. F. Morales, and R. Poghossian, *Instantons on Quivers and Orientifolds*, *JHEP* **0410:037,2004** (JHEP 0410:037,2004) [hep-th/0408090].
- [58] D. Gaiotto, *Asymptotically free $N=2$ theories and irregular conformal blocks*, 0908.0307.
- [59] N. Wyllard, *A_{N-1} conformal Toda field theory correlation functions from conformal $N=2$ $SU(N)$ quiver gauge theories*, *JHEP* **0911:002,2009** (July, 2009) [0907.2189].
- [60] P. Bowcock and G. M. T. Watts, *Null vectors, 3-point and 4-point functions in conformal field theory*, *Theor.Math.Phys.* 500-508 **Theor.Math.Phys.** **500-508** (1994 500-508) *Theor.Math.Phys.* 98(1994)350–356; *Teor.Mat.Fiz.* 98(1994) 500–508, [hep-th/9309146].
- [61] V. A. Fateev and A. V. Litvinov, *On differential equation on four-point correlation function in the Conformal Toda Field Theory*, *JETP Lett.* 728-732 **JETP** **728-732** (2005 728-732) *JETPLett.* 81(2005)594–598; *PismaZh.Eksp.Teor.Fiz.* 81(2005) 728–732, [hep-th/0505120].
- [62] V. A. Fateev and A. V. Litvinov, *Correlation functions in conformal Toda field theory I*, *JHEP* **0711:002,2007** (Sept., 2007) [0709.3806].
- [63] N. A. Nekrasov and S. L. Shatashvili, *Quantization of Integrable Systems and Four Dimensional Gauge Theories*, 0908.4052.
- [64] Sergei Gukov and Edward Witten, *Rigid Surface Operators*, 0804.1561.
- [65] D. Gaiotto, *Surface Operators in $N=2$ 4d Gauge Theories*, 0911.1316.

- [66] A. Marshakov, A. Mironov, and A. Morozov, *On AGT Relations with Surface Operator Insertion and Stationary Limit of Beta-Ensembles*, *J.Geom.Phys.* **61:1203-1222,2011** (Nov., 2010) [1011.4491].
- [67] K. Maruyoshi and M. Taki, *Deformed Prepotential, Quantum Integrable System and Liouville Field Theory*, 1006.4505.
- [68] A. Mironov and A. Morozov, *Nekrasov Functions from Exact BS Periods: the Case of $SU(N)$* , *J.Phys.A* **43:195401,2010** (Nov., 2009) [0911.2396].
- [69] A. Mironov and A. Morozov, *Nekrasov Functions and Exact Bohr-Sommerfeld Integrals*, *JHEP* **04** (Oct., 2009) 040, [0910.5670].
- [70] A. Brini, M. Marino, and S. Stevan, *The uses of the refined matrix model recursion*, 1010.1210.
- [71] H. Awata, H. Fuji, H. Kanno, M. Manabe, and Y. Yamada, *Localization with a Surface Operator, Irregular Conformal Blocks and Open Topological String*, 1008.0574.
- [72] M. Taki, *Surface Operator, Bubbling Calabi-Yau and AGT Relation*, *JHEP* **1107:047,2011** (July, 2010) [1007.2524].
- [73] U. Bruzzo, W. y. Chuang, D. E. Diaconescu, M. Jardim, G. Pan, and Y. Zhang, *D-branes, surface operators, and ADHM quiver representations*, 1012.1826.
- [74] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, *Moduli Space of Non-Abelian Vortices*, *Phys.Rev.Lett.* **96:161601,2006** (Phys.Rev.Lett.96:161601,2006) [hep-th/0511088].
- [75] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai, *Solitons in the Higgs phase – the moduli matrix approach*, *J.Phys.A* **39:R315-R392,2006** (J.Phys.A39:R315-R392,2006) [hep-th/0602170].
- [76] U. Bruzzo, F. Fucito, J. F. Morales, and A. Tanzini, *Multi-Instanton Calculus and Equivariant Cohomology*, *JHEP* **0305:054,2003** (JHEP 0305:054,2003) [hep-th/0211108].
- [77] A. Mironov and A. Morozov, *On AGT relation in the case of $U(3)$* , *Nucl.Phys.B* **825:1-37,2010** (Aug., 2009) [0908.2569].
- [78] A. Mironov and A. Morozov, *The Power of Nekrasov Functions*, *Phys.Lett.B* **680:188-194,2009** (Aug., 2009) [0908.2190].

- [79] R. Flume and R. Poghossian, *An Algorithm for the Microscopic Evaluation of the Coefficients of the Seiberg-Witten Prepotential*, *Int.J.Mod.Phys. A* **18** (2003) 2541, [hep-th/0208176].
- [80] A. A. Gerasimov and D. R. Lebedev, *On topological field theory representation of higher analogs of classical special functions*, *JHEP* **1109:076,2011** (Nov., 2010) [1011.0403].
- [81] H. Awata and Y. Yamada, *Five-dimensional AGT Conjecture and the Deformed Virasoro Algebra*, *JHEP* **1001:125,2010** (Oct., 2009) [0910.4431].
- [82] V. A. Fateev, A. V. Litvinov, A. Neveu, and E. Onofri, *Differential equation for four-point correlation function in Liouville field theory and elliptic four-point conformal blocks*, *J.Phys.A* **42:304011,2009** (Feb., 2009) [0902.1331].
- [83] N. Drukker, D. Gaiotto, and J. Gomis, *The Virtue of Defects in 4D Gauge Theories and 2D CFTs*, 1003.1112.
- [84] Y. Nakayama, *Liouville Field Theory – A decade after the revolution*, *Int.J.Mod.Phys.A* **19:2771-2930,2004** (Int.J.Mod.Phys.A19:2771-2930,2004) [hep-th/0402009].
- [85] T. Fukuda and K. Hosomichi, *Super Liouville Theory with Boundary*, *Nucl.Phys.B* **635:215-254,2002** (Nucl.Phys.B635:215-254,2002) [hep-th/0202032].
- [86] D. Chorazkiewicz, L. Hadasz, and Z. Jaskolski, *Braiding properties of the $N=1$ super-conformal blocks (Ramond sector)*, 1108.2355.
- [87] F. Fucito, J. F. Morales, and R. Poghossian, *Multi instanton calculus on ALE spaces*, *Nucl.Phys. B* **703** (2004) 518–536, [hep-th/0406243].
- [88] J. M. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*, *Adv.Theor.Math.Phys.* **2:231-252,1998** (Adv.Theor.Math.Phys.2:231-252,1998) [hep-th/9711200].
- [89] D. Gaiotto and J. Maldacena, *The gravity duals of $N=2$ superconformal field theories*, 0904.4466.
- [90] A. Gadde, E. Pomoni, and L. Rastelli, *The Veneziano Limit of $N=2$ Superconformal QCD: Towards the String Dual of $N=2$ $SU(N_c)$ SYM with $N_f = 2N_c$* , 0912.4918.
- [91] E. Frenkel and D. Gaitsgory, *Local geometric Langlands correspondence and affine Kac-Moody algebras*, math/0508382.

- [92] A. Kapustin and E. Witten, *Electric-Magnetic Duality And The Geometric Langlands Program*, hep-th/0604151.
- [93] S. Gukov and E. Witten, *Gauge Theory, Ramification, And The Geometric Langlands Program*, hep-th/0612073.
- [94] J. Teschner, *Quantization of the Hitchin moduli spaces, Liouville theory, and the geometric Langlands correspondence I*, 1005.2846.
- [95] S. Gukov, *Surface Operators and Knot Homologies*, 0706.2369.
- [96] E. Witten, *Khovanov Homology And Gauge Theory*, 1108.3103.